PERGAMON

Neural Networks 12 (1999) 837–850

Neural Networks

www.elsevier.com/locate/neunet

# Properties of learning of a Fuzzy ART Variant

M. Georgiopoulos[a,*], I. Dagher[a], G.L. Heileman[b], G. Bebis[c]

[a]Department of Electrical and Computer Engineering, University of Central Florida, Orlando, FL 32816, USA
[b]Department of Electrical and Computer Engineering, University of New Mexico, Albuquerque, NM 87131, USA
[c]Department of Computer Science, University of Nevada Reno, Reno, NV 89557, USA

Received 18 November 1996; received in revised form 26 March 1999; accepted 26 March 1999

**Abstract**

This paper discusses a variation of the Fuzzy ART algorithm referred to as the *Fuzzy ART Variant*. The Fuzzy ART Variant is a Fuzzy ART algorithm that uses a very large choice parameter value. Based on the geometrical interpretation of the weights in Fuzzy ART, useful properties of learning associated with the Fuzzy ART Variant are presented and proven. One of these properties establishes an upper bound on the number of list presentations required by the Fuzzy ART Variant to learn an arbitrary list of input patterns. This bound is small and demonstrates the short-training time property of the Fuzzy ART Variant. Through simulation, it is shown that the Fuzzy ART Variant is as good a clustering algorithm as a Fuzzy ART algorithm that uses typical (i.e. small) values for the choice parameter. © 1999 Elsevier Science Ltd. All rights reserved.

*Keywords:* Neural network; Unsupervised learning; Supervised learning; Clustering; Adaptive resonance theory

## 1. Introduction

Adaptive resonance theory was developed by Grossberg (1976), and a large number of the ART architectures have been introduced in the last 10 years (e.g. Carpenter & Grossberg, 1987a; Carpenter & Grossberg, 1987b; Carpenter & Grossberg, 1990; Carpenter, Grossberg & Reynolds, 1991a; Carpenter, Grossberg & Reynolds, 1991b; Carpenter, Grossberg, Markuzon, Reynolds & Rosen, 1992; Carpenter & Gjaja, 1994; Carpenter & Ross, 1995; Carpenter & Markuzon, 1998; Healy, Caudell & Smith, 1993; Marriott & Harrison, 1995; Tan, 1995; Williamson, 1996) A major separation among all of these architectures is based on whether the learning applied is unsupervised or supervised. Unsupervised learning is implemented when a collection of input patterns needs to be appropriately clustered into categories, while supervised learning is utilized when a mapping needs to be learned between inputs and corresponding output patterns. A prominent member of the class of unsupervised ART architectures is Fuzzy ART (Carpenter et al., 1991b), which is capable of clustering arbitrary collections of arbitrarily complex analog input patterns. Our focus in this paper is Fuzzy ART and its associated properties of learning.

Properties of learning for Fuzzy ART have already been reported in the literature (Carpenter et al., 1991b; Huang et al., 1995). Most of these properties pertain to a Fuzzy ART network whose choice parameter is small. In particular, one of our favorite properties of learning in Fuzzy ART (i.e. its short training time) has been reported only for small values of the choice parameter. The Fuzzy ART algorithm was initially introduced for values of the choice parameter ranging over the interval $(0, \infty)$ (Carpenter et al., 1991b). It is therefore an issue of intellectual curiosity and theoretical importance as to how these learning properties change as we move from the domain of small choice parameter values to that of large choice parameter values. Some work towards this goal has appeared in the literature. For example, Georgiopoulos, Fernlund, Bebis and Heileman (1996) demonstrated the "Order of Search" property of learning in Fuzzy ART. The "Order of Search" property identifies the order according to which nodes in the category representation field of Fuzzy ART are chosen. In particular, three distinct orders of search were identified for three different ranges of the choice parameter value: (i) choice parameter small ($\alpha \to 0$), (ii) choice parameter large ($\alpha \to \infty$), and (iii) choice parameter of intermediate value ($0 < \alpha < \infty$). This paper extends the work of Georgiopoulos et al. (1996) to other properties of learning. Specifically, we investigate the short-training time property of Fuzzy ART, assuming the choice parameter is very large. For simplicity, this Fuzzy ART network is referred to as the *Fuzzy ART Variant*,

* Corresponding author. Tel.: + 1-407-823-5338; fax: + 1-407-823-5835.
*E-mail address:* mng@ece.engr.ucf.edu (M. Georgiopoulos)

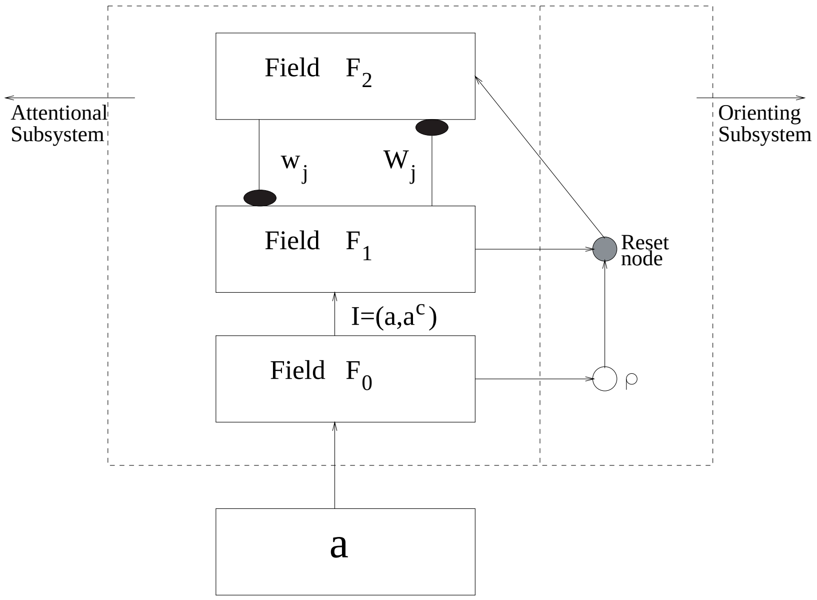


Fig. 1. A block diagram of the Fuzzy ART architecture.

despite the fact that it is simply Fuzzy ART with large values of the choice parameter. In the process of verifying the short-training time property of Fuzzy ART Variant, other useful properties of learning of the Fuzzy ART Variant were discovered. It should also be mentioned that the aforementioned Fuzzy ART Variant algorithm was described by Carpenter and Gjaja (1994).

The organization of the paper is as follows. In Section 2, the specifics of the Fuzzy ART network that are pertinent to this paper are briefly discussed. In Section 3 the Fuzzy ART Variant is introduced, and some of the differences between Fuzzy ART with small choice parameter values and the Fuzzy ART Variant are emphasized. In Section 4 three properties of learning in the Fuzzy ART Variant are proven and discussed, including the short-training time property. In Section 5 it is demonstrated through simulations that the Fuzzy ART Variant is as good a clustering algorithm as a Fuzzy ART network with small values for the choice parameter. Finally, in Section 6 a short review and concluding remarks are provided.

## 2. Fuzzy ART

### 2.1. Fuzzy ART architecture

The Fuzzy ART neural network architecture is shown in Fig. 1. It consists of two subsystems, *the attentional subsystem*, and the *orienting subsystem*. The attentional subsystem consists of two fields of nodes denoted $F_1$ and $F_2$. The $F_1$ field is called the *input field* because input patterns are applied to it. The $F_2$ field is called the *category or class representation field* because it is the field where category representations are formed. These category representations represent the clusters to which the input patterns belong. The orienting subsystem consists of a single node (called the *reset node*), which accepts inputs from the $F_1$ field, the $F_2$ field (not shown in Fig. 1), and the input pattern applied across the $F_1$ field. The output of the reset node affects the nodes of the $F_2$ field.

Some preprocessing of the input patterns of the pattern-clustering task takes place before they are presented to Fuzzy ART. The first preprocessing stage takes as input an $M$-dimensional input pattern from the pattern clustering task and transforms it into an output vector $\mathbf{a} = (a_1, \ldots, a_M)$, whose every component lies in the interval [0,1] (i.e. $0 \le a_i \le 1$ for $1 \le i \le M$). The second preprocessing stage accepts as an input the output $\mathbf{a}$ of the first preprocessing stage and produces an output vector $\mathbf{I}$, such that

$$\mathbf{I} = (\mathbf{a}, \mathbf{a}^c) = (a_1, \ldots, a_M, a_1^c, \ldots, a_M^c), \tag{1}$$

where

$$a_i^c = 1 - a_i; \qquad 1 \le i \le M. \tag{2}$$

The above transformation is called *complement coding*. The complement coding operation is performed in Fuzzy ART at a preprocessor field designated by $F_0$ (see Fig. 1). From now on, the vector $\mathbf{I}$ will be referred to as the *input pattern*. Each category $j$ ($1 \le j \le N$) in the category representation layer corresponds to a vector $\mathbf{w}_j = (w_{j1}, \ldots, w_{j,2M})$ of adaptive weights. The initial values for these weights are chosen to be equal to $w_{j1} = \ldots = w_{j,2M} = 1$, and a category with these weights is said to be *uncommitted*. Initial values for these weights may be taken greater than one. Larger weights bias the system against the selection of uncommitted nodes, leading to deeper searches of previously coded categories. After a category is chosen to represent an input pattern it is referred to as a *committed category* or *committed node*. Prior to this point, it is an *uncommitted category* or *uncommitted node*. It is worth noting that the Fuzzy ART weight vector $\mathbf{w}_j$ subsumes both the bottom–up and top–down weight vectors of Fuzzy ART.

The training phase of Fuzzy ART works as follows: given a list of input patterns, designated as $\mathbf{I}^1, \mathbf{I}^2, \ldots, \mathbf{I}^P$, we want to train Fuzzy ART to cluster these input patterns into different categories. Obviously, patterns that are similar to each other are expected to be clustered in the same category by Fuzzy ART. In order to achieve the aforementioned goal, the training list is repeatedly presented to the Fuzzy ART architecture. That is, $\mathbf{I}^1$ is presented first, then $\mathbf{I}^2$, and eventually $\mathbf{I}^P$; this corresponds to one *list presentation*. Then, if it is necessary, $\mathbf{I}^1, \mathbf{I}^2, \ldots, \mathbf{I}^P$ are presented again. The training list is presented as many times as is necessary for Fuzzy ART to cluster the input patterns. The clustering task is considered accomplished (i.e. learning is complete) if the weights in the Fuzzy ART architecture do not change during a list presentation. The above training scenario is called *off-line training*.

Before discussing in more detail the training phase of Fuzzy ART, let us elaborate on the Fuzzy ART parameters involved in its training phase. The parameter $\alpha$, called the *choice parameter*, takes values in the interval $(0,\infty)$. It affects the bottom–up inputs that are produced at the $F_2$ nodes due to the presentation of an input pattern at $F_1$. The parameter $\rho$ is called the *vigilance parameter* and it takes values in the interval [0,1]. Small values of $\rho$ result in coarse clustering of the input patterns, while large values of $\rho$ result in fine clustering of the input patterns. The parameter $N$ corresponds to the number of committed nodes during the training phase of Fuzzy ART. During the training phase Fuzzy ART operates over all of the committed nodes along with a single uncommitted node. A committed node (category) in $F_2$ is a node that has coded at least one input pattern. An uncommitted node (category) is a node that is not committed.

The step-by-step implementation of the off-line training phase of Fuzzy ART is presented below. The Fuzzy ART network parameters $\alpha$, $\rho$, and $N$ are chosen at the beginning of the training phase; $\alpha$ is chosen from $(0,\infty)$ but typically small, $\rho$ is chosen from [0,1] based on the fineness of the clusters we want to create, and obviously $N = 0$. Further, the initial components of the weight vector corresponding to the first uncommitted node (i.e. $w_{1i}(0)$s $1 \le i \le 2M$) are chosen to be equal to one. The value of the pattern index $r$ is initialized to 1. For compactness of the presentation the definitions of the various functions that appear in the step-by-step implementation of the training phase will be provided after the step-by-step description is completed.

### 2.1.1. *Off-line training phase of Fuzzy ART*

1. Choose the $r$th input pattern from the training list.
2. Calculate the bottom–up inputs to all the $N + 1$ nodes in $F_2$ due to the presentation of the $r$th input pattern. When calculating bottom–up inputs consider all the committed nodes and the uncommitted node. These bottom–up inputs are calculated according to the following equation.

$$T_j(\mathbf{I}^r) = \begin{cases} \dfrac{M}{\alpha + 2M} & \text{if } j \text{ is the uncommitted node} \\ \dfrac{|\mathbf{I}^r \wedge \mathbf{w}_j|}{\alpha + |\mathbf{w}_j|} & \text{if } j \text{ is a committed node} \end{cases}. \tag{3}$$

3. Choose the node in $F_2$ that receives the maximum bottom–up input from $F_1$. Assume that this node has index $j_{\max}$. Check to see whether this node satisfies the vigilance criterion. Three cases are now distinguished:

   (a) If node $j_{\max}$ is the uncommitted node it satisfies the vigilance criterion. Increase the parameter $N$ by one. This way a new uncommitted node in $F_2$ is introduced, and its initial weight vector is chosen to be the "all-ones" vector. Go to Step 4.
   (b) If node $j_{\max}$ is a committed node, and it satisfies the vigilance criterion, go to Step 4. A committed node $j_{\max}$ satisfies the vigilance criterion if

$$\frac{|\mathbf{I}^r \wedge \mathbf{w}_{j_{\max}}|}{|\mathbf{I}^r|} \ge \rho \tag{4}$$

   (c) If node $j_{\max}$ does not satisfy the vigilance criterion, disqualify this node by setting $T_{j_{\max}}(\mathbf{I}^r) = -1$, and go to the beginning of Step 3.

4. The weights associated with node $j_{\max}$ are modified according to the following equation:

$$\mathbf{w}_{j_{\max}} = \mathbf{w}_{j_{\max}} \wedge \mathbf{I}^r. \tag{5}$$

   If this is the last input pattern in the training list go to Step 5. Otherwise, go to Step 1 to present the next in sequence input pattern by increasing the index $r$ by one.
5. After all the patterns have been presented consider two cases:

   (a) In the previous list presentation at least one component of the weight vectors has been changed. In this case, go to Step 1, and present the first input pattern, by resetting the index $r$ to the value 1.
   (b) In the previous list presentation no weight changes occurred. In this case, the learning process is considered complete.

In Step 5(b) mentioned above it is implied that there exists a finite-valued list presentation at which no weight changes occur. Unfortunately, no theoretical result exists to justify this claim for all values of the choice parameter $\alpha$. For very small $\alpha$ parameter values this claim is valid because learning will be over in one list presentation (Carpenter et al., 1991b). For values of the parameter $\alpha$

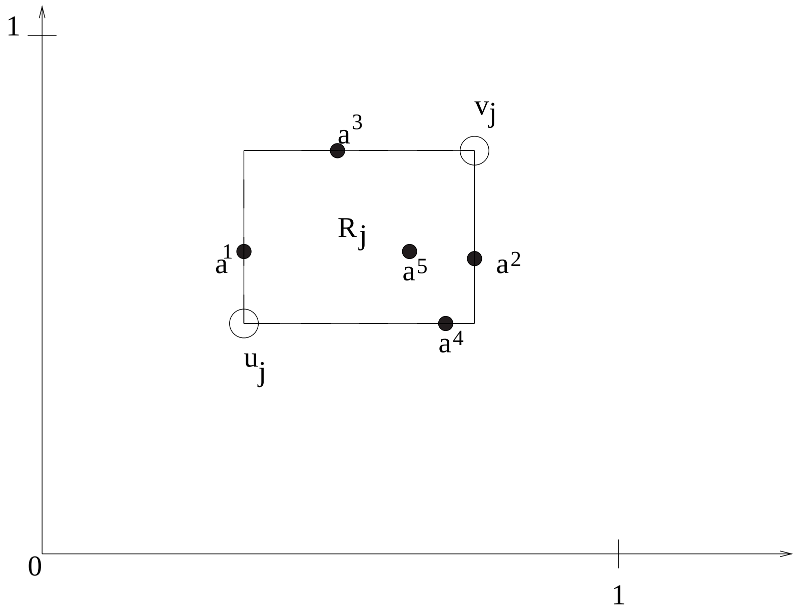


Fig. 2. The hyper-rectangle (rectangle) which has coded the input patterns $\mathbf{I}^1 = (\mathbf{a}^1, (\mathbf{a}^1)^c)$, $\mathbf{I}^2 = (\mathbf{a}^2, (\mathbf{a}^2)^c)$, $\mathbf{I}^3 = (\mathbf{a}^3, (\mathbf{a}^3)^c)$, $\mathbf{I}^4 = (\mathbf{a}^4, (\mathbf{a}^4)^c)$, and $\mathbf{I}^5 = (\mathbf{a}^5, (\mathbf{a}^5)^c)$.

that are not small some specialized results are discussed in Huang et al. (1995).

In the definition of the bottom–up inputs produced in $F_2$ due to the presentation of the input pattern $\mathbf{I}^r$ (see Eq. (3)) the “fuzzy-min” ($\wedge$) operation of two vectors, $\mathbf{I}^r$ and $\mathbf{w}_j$, is introduced. The fuzzy-min operation of two vectors $\mathbf{I}^r$ and $\mathbf{w}_j$ is a vector whose components are equal to the minimum of the corresponding components of $\mathbf{I}^r$ and $\mathbf{w}_j$. This same operation was used in the calculation of the vigilance ratio (see Eq. (4)) and in the updates of the weights (see Eq. (5)). Also, in the definition of the bottom–up inputs produced in $F_2$ due to the presentation of the input pattern $\mathbf{I}^r$ (see Eq. (3)) the operation $|\cdot|$ (e.g., $|\mathbf{w}_j|$) is introduced, that stands for the size of a vector. The size of a vector is defined to be the sum of its components. This operation was also used in the definition of the vigilance ratio in Eq. (4).

It has been shown in Carpenter et al. (1991b) that the weight vectors (i.e. the $\mathbf{w}_j$’s), corresponding to committed nodes in Fuzzy ART, have a geometrical interpretation. That is, $\mathbf{w}_j$ can be expressed as ($\mathbf{u}_j$, $(\mathbf{v}_j)^c$), where $\mathbf{u}_j$ is the lower endpoint, and $\mathbf{v}_j$ the upper endpoint of a hyper-rectangle. This hyper-rectangle lies in the $M$-dimensional space and includes all the input patterns that have chosen and were coded by node $j$. Using this representation the input pattern $\mathbf{I} = (\mathbf{a}, \mathbf{a}^c)$ can also be thought of as a hyper-rectangle with lower endpoint $\mathbf{a}$ and upper endpoint $\mathbf{a}$ (that is a hyper-rectangle of size 0). To visualize this hyper-rectangle notion the hyper-rectangle $R_j$ is shown (see Fig. 2), with endpoints $\mathbf{u}_j$ and $\mathbf{v}_j$, corresponding to the weight vector $\mathbf{w}_j$, which has coded the input patterns $\mathbf{I}^1 = (\mathbf{a}^1, (\mathbf{a}^1)^c)$, $\mathbf{I}^2 = (\mathbf{a}^2, (\mathbf{a}^2)^c)$, $\mathbf{I}^3 = (\mathbf{a}^3, (\mathbf{a}^3)^c)$, $\mathbf{I}^4 = (\mathbf{a}^4, (\mathbf{a}^4)^c)$, and $\mathbf{I}^5 = (\mathbf{a}^5, (\mathbf{a}^5)^c)$. In Fig. 2, the $\mathbf{I}$’s are 4-D and the $\mathbf{a}$’s, $\mathbf{u}_j$ and $\mathbf{v}_j$ are 2-D. As most of our illustrations from now on will be in the 2-D space hyper-rectangles are referred to as rectangles.

## 3. The Fuzzy ART Variant algorithm

As it was emphasized in the Introduction, the primary focus in this paper is the Fuzzy ART algorithm with a very large choice parameter value $\alpha$ (i.e. $\alpha \longrightarrow \infty$). One might question this choice, as when $\alpha$ is large Fuzzy ART has the tendency to choose uncommitted nodes over existing committed nodes (see Eq. (3)). This way we may end up with a Fuzzy ART algorithm that does not perform useful clustering as every input pattern from the training list forms its own cluster. As it was mentioned in the Fuzzy ART paper though (see Carpenter et al., 1991b) initial values of the weight components corresponding to uncommitted nodes may be taken larger than one. Taking it to the extreme these initial values of the weight components of the uncommitted nodes can be chosen to be very large so that the bottom–up inputs to the uncommitted nodes are approximately equal to zero. By choosing, at the same time, a very large value for the choice parameter, the bottom–up inputs to uncommitted nodes will still be approximately zero, and the bottom–up inputs to a committed node will be proportional to the size of the “fuzzy-min” of the input pattern vector and the weight vector corresponding to this node. The Fuzzy ART algorithm with very large values of the initial components for the uncommitted weights, and very large value of the choice parameter is called *Fuzzy ART Variant*.

The step-by-step implementation of the off-line training phase of Fuzzy ART Variant algorithm is presented below.

The Fuzzy ART Variant network parameters, $\alpha$, $\rho$, and $N$, are chosen at the beginning of the training phase; $\alpha$ is chosen from $(0,\infty)$ but typically very large, $\rho$ is chosen from [0, 1] based on the fineness of the clusters we want to create, and obviously $N = 0$. Further, the initial weights of the first uncommitted node (i.e. the $w_{1i}(0)$'s) are chosen to be very large too. The value of the pattern index $r$ is initialized to 1.

### 3.1. Off-line training phase of Fuzzy ART Variant

The training algorithm of the Fuzzy ART Variant is the same as the Fuzzy ART algorithm (see Section 2.1), except Step 2, which is replaced by Step 2 (Fuzzy ART Variant) as follows:

*Step 2 (Fuzzy ART Variant)*: Calculate the bottom–up inputs to all the $N + 1$ nodes in $F_2$ due to the presentation of the $r$th input pattern. When calculating bottom–up inputs consider all the committed nodes and the uncommitted node. These bottom–up inputs are calculated according to the following equation:

$$T_j(\mathbf{I}^r) = \begin{cases} 0 & \text{if } j \text{ is the uncommitted node} \\ \lim_{\alpha\to\infty} \dfrac{|\mathbf{I}^r \wedge \mathbf{w}_j|}{\alpha + |\mathbf{w}_j|} \approx \dfrac{1}{\alpha}|\mathbf{I}^r \wedge \mathbf{w}_j| & \text{if } j \text{ is a committed node} \end{cases}. \tag{6}$$

Note that $|\mathbf{w}_j|$ stands for the size of weight vector $\mathbf{w}_j$, where the size of a vector is defined to be equal to the sum of its components.

One way of understanding the differences between Fuzzy ART (small values of the choice parameter) and Fuzzy ART Variant (large values of the choice parameter) is by reporting the order according to which nodes in $F_2$ are chosen for these two algorithms. This topic has been extensively investigated by Carpenter and Grossberg (1987b) for ART1, and by Georgiopoulos et al. (1996) for Fuzzy ART. Two of the results discussed in Georgiopoulos et al. (1996) (denoted Results A and B) are reproduced here to illustrate some of the differences of Fuzzy ART and Fuzzy ART Variant.

**Result A.** If an input pattern **I** is presented to a Fuzzy ART architecture with small $\alpha$ parameter values (i.e. $\alpha$ close to zero) and:

1. **I** is inside rectangles $R_{j_1}^{\text{old}}$ and $R_{j_2}^{\text{old}}$, then **I** will choose first the rectangle of the smallest size.
2. **I** is outside rectangle $R_{j_1}^{\text{old}}$ and inside rectangle $R_{j_2}^{\text{old}}$, then **I** will choose first rectangle $R_{j_2}^{\text{old}}$.
3. **I** is outside rectangles $R_{j_1}^{\text{old}}$ and $R_{j_2}^{\text{old}}$, then **I** will choose first rectangle $R_{j_1}^{\text{old}}$, iff

$$\text{dis}(\mathbf{I}, R_{j_1}^{\text{old}}) < \frac{M - |R_{j_1}^{\text{old}}|}{M - |R_{j_1}^{\text{old}}|}\text{dis}(\mathbf{I}, R_{j_2}^{\text{old}}) \tag{7}$$

**Result B.** If an input pattern **I** is presented to a Fuzzy ART architecture with large $\alpha$ parameter values (i.e. $\alpha$ approaching $\infty$), and:

1. **I** is inside rectangles $R_{j_1}^{\text{old}}$ and $R_{j_2}^{\text{old}}$, then **I** will choose first the rectangle of the smallest size.
2. **I** is outside rectangle $R_{j_1}^{\text{old}}$ and inside rectangle $R_{j_2}^{\text{old}}$, then **I** will choose first rectangle $R_{j_1}^{\text{old}}$, iff

$$|R_{j_1}^{\text{new}}| < |R_{j_2}^{\text{old}}| \tag{8}$$

3. **I** is outside rectangles $R_{j_1}^{\text{old}}$ and $R_{j_2}^{\text{old}}$, then **I** will choose first rectangle $R_{j_1}^{\text{old}}$, iff

$$|R_{j_1}^{\text{new}}| < |R_{j_2}^{\text{new}}|.$$

Note that $R_j^{\text{old}}$ is the rectangle corresponding to node $j$ of field $F_2$ prior to the presentation of input pattern **I** at the field $F_1$. Also, $R_j^{\text{new}}$ is the new rectangle corresponding to node $j$ that would have been created if pattern **I** were to choose and be coded by node $j$. Note that if the input pattern **I** is inside rectangle $R_j^{\text{old}}$, then $R_j^{\text{old}} = R_j^{\text{new}}$; otherwise $R_j^{\text{old}} \neq R_j^{\text{new}}$ and, in particular, $R_j^{\text{new}}$ includes $R_j^{\text{old}}$. Further, $|R_j|$ stands for the size of rectangle $R_j$ corresponding to node $j$ in the field $F_2$; the size of a rectangle $R_j$ is defined to be the city block distance between the endpoints $\mathbf{u}_j$ and $\mathbf{v}_j$ of this rectangle. Finally, the distance of an input pattern $\mathbf{I} = (\mathbf{a}, \mathbf{a}^c)$ from a rectangle $R_j$, which does not contain **I**, is defined to be the minimum of the city block distances of **a** from points that belong to the boundary of $R_j$.

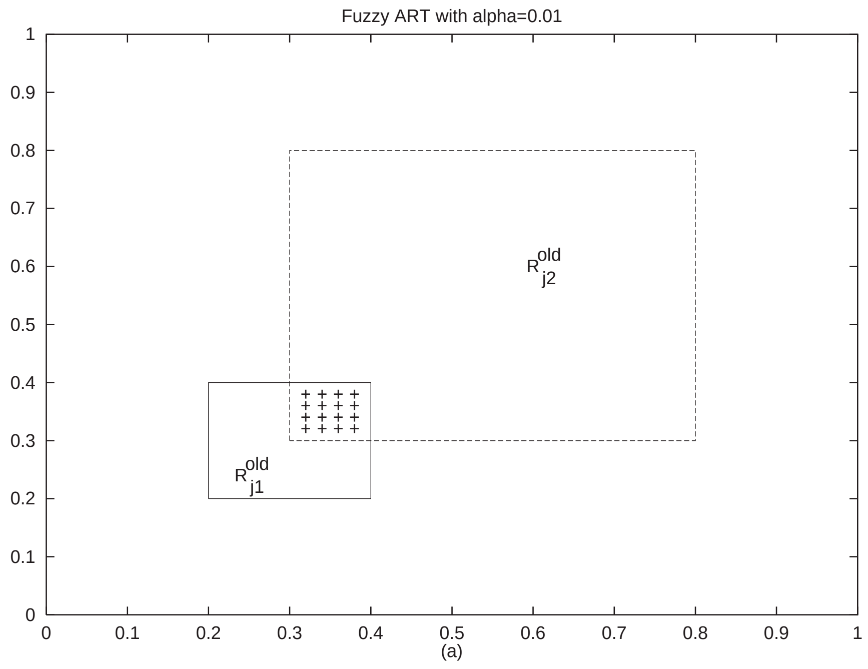
To visualize the similarities and differences between the Fuzzy ART choices made when $\alpha$ is small, and when $\alpha$ is large (Fuzzy ART Variant) the consequences of Results A and B are depicted in Fig. 3 for three example cases.

## 4. Properties of learning of the Fuzzy ART variant

In this section three properties of learning of the Fuzzy ART Variant are reported. Further, their importance is emphasized, their proofs are provided, and finally an example case is discussed. These properties of learning are referred to as Results 1–3.

### 4.1. Results

Result 1 tells us that, during the Fuzzy ART Variant training, the identity of the node with the largest rectangle does not change after the first list presentation, the identity



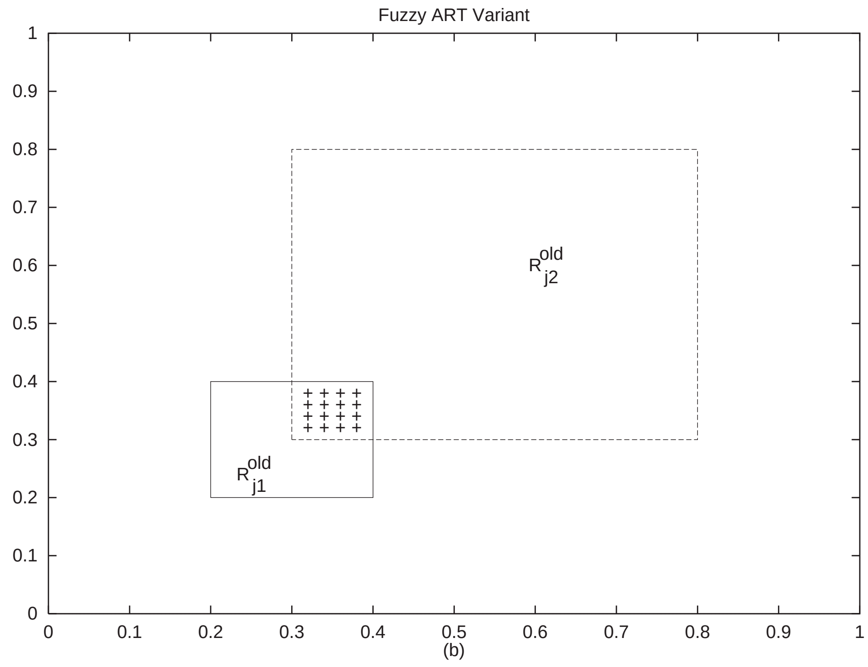
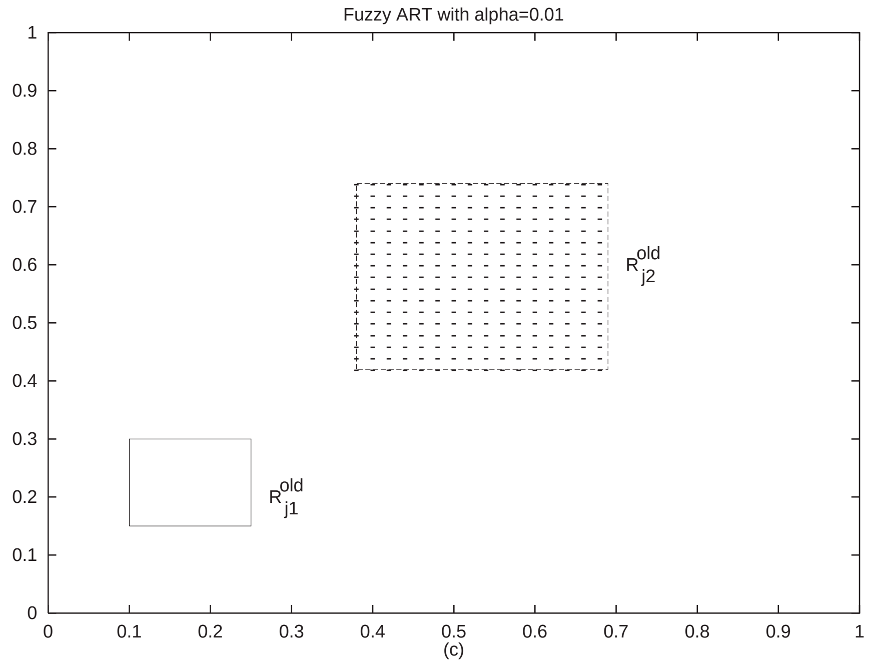


Fig. 3. Illustration of the Order-of-Search properties (Results A and B) for Fuzzy ART (small values of the choice parameter) and Fuzzy ART Variant. (**a**,**b**): Pattern **I** is inside rectangles $R^{old}_{j_1}$, and $R^{old}_{j_2}$, where $|R^{old}_{j_1}| < |R^{old}_{j_2}|$; the "+" signs indicate patterns **I** that choose first rectangle $R^{old}_{j_1}$ and the "−" signs indicate patterns **I** that choose first rectangle $R^{old}_{j_2}$. (**c**,**d**): Pattern **I** is outside rectangle $R^{old}_{j_1}$, and inside rectangle $R^{old}_{j_2}$; the "+" signs indicate patterns **I** that choose first rectangle $R^{old}_{j_1}$ and the "− " signs indicate patterns **I** that choose first rectangle $R^{old}_{j_2}$. (**e**,**f**): Pattern **I** is outside rectangles $R^{old}_{j_1}$ and $R^{old}_{j_2}$; the "+" signs indicate patterns **I** that choose first rectangle $R^{old}_{j_1}$ and the "−" signs indicate patterns **I** that choose first rectangle $R^{old}_{j_2}$.

of the node with the second largest rectangle does not change after the second list presentation, and so on. Result 1 also tells us that, during the Fuzzy ART Variant training, the size of the largest rectangle does not change after the first list presentation, the size of the second largest rectangle does not change after the second list presentation, and so on.

**Result 1.** Consider the off-line training of a list of $P$ input patterns using the Fuzzy ART Variant algorithm. Assume that after the first list presentation the Fuzzy ART Variant has created $N$ categories in $F_2$. Designate by $j_n(t)(1 \leq n \leq N; t \geq 1)$ the index of the node with the $n$th largest rectangle immediately after the end of the $t$th list



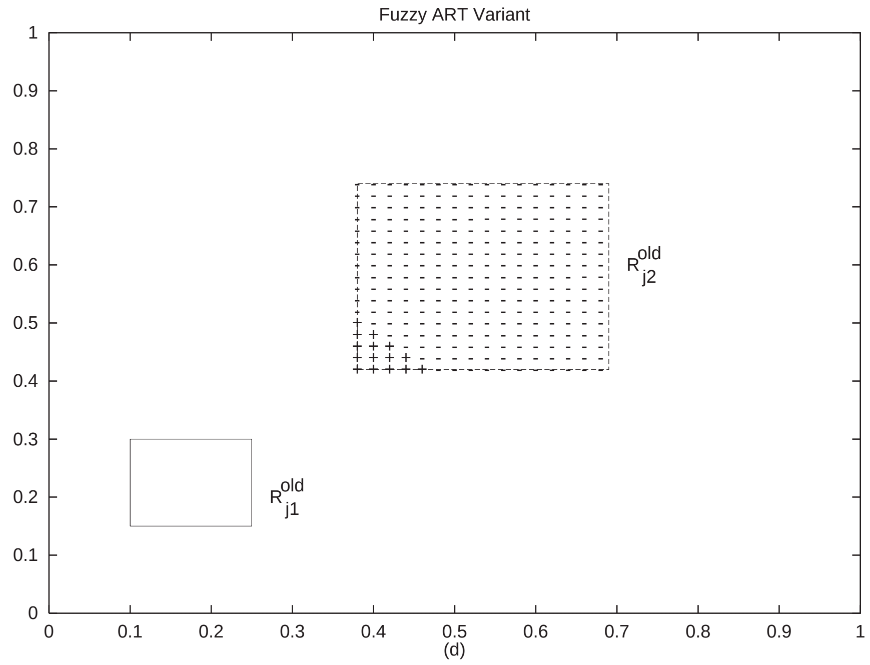
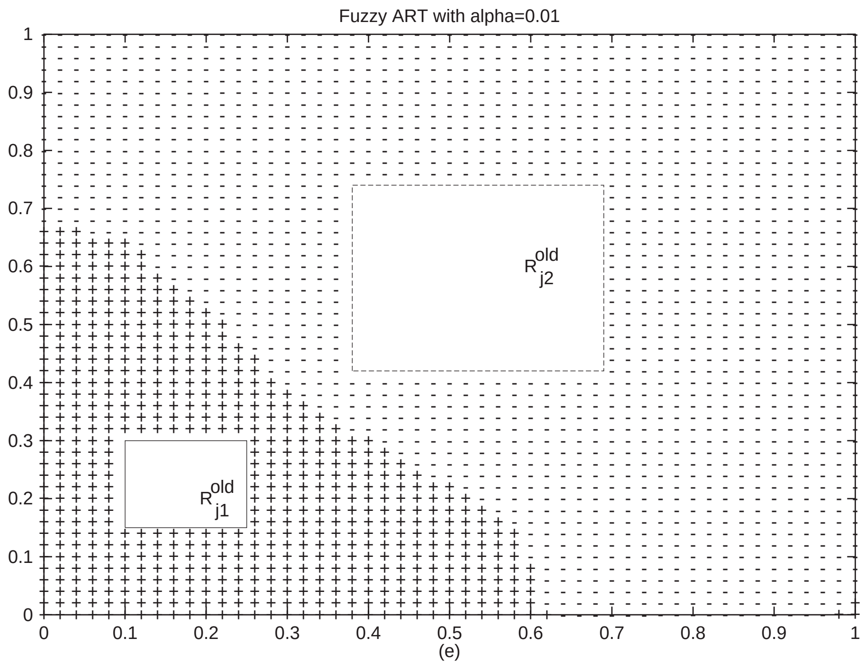


Fig. 3. (*continued*)

presentation. Then,

$$j_n(t) = j_n(n) \qquad \text{for } 1 \le n \le N; t \ge n+1 \qquad (10)$$
$$|R_{j_n(t)}| = |R_{j_n(n)}|$$

Result 2 tells us that, during the Fuzzy ART Variant training, patterns that are coded by the largest rectangle in the second list presentation do not need to be presented to the Fuzzy ART architecture again, patterns that are coded by the second largest rectangle in the third list presentation do not need to be presented to the Fuzzy ART Variant again, and so on. Result 2 is useful because it allows us to eliminate patterns from the training list that do not affect the Fuzzy ART Variant learning process. This way the learning process can be made to be less computationally intensive.



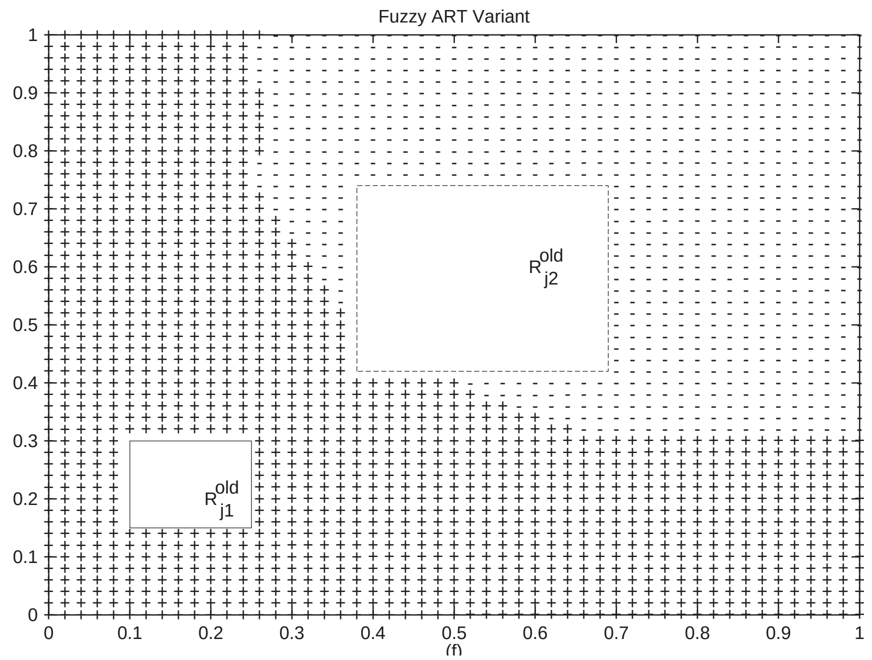


Fig. 3. (*continued*)

**Result 2.** Consider the off-line training of a list of input patterns using the Fuzzy ART Variant algorithm. Assume that after the first list presentation the Fuzzy ART Variant algorithm has created $N$ categories in $F_2$. Designate by $j_n(t)(1 \leq n \leq N; t \geq 1)$ the index of the node with $n$-th largest rectangle immediately after the end of the $t$-th list presentation. Let $S_n(1 \leq n)$ denote the set of training patterns that choose and are coded by node $j_n(n)$ in the $(n+1)$th list presentation. Then, the patterns of collection $S_n$ will always be coded by node $j_n(n)$ in list presentations $\geq n+2$.

Result 3 is important because it predicts an upper bound for the number of list presentations required by the Fuzzy ART Variant to learn a list of input patterns. In order to identify this upper bound, it suffices to present once the training list through the Fuzzy ART network using the Fuzzy ART

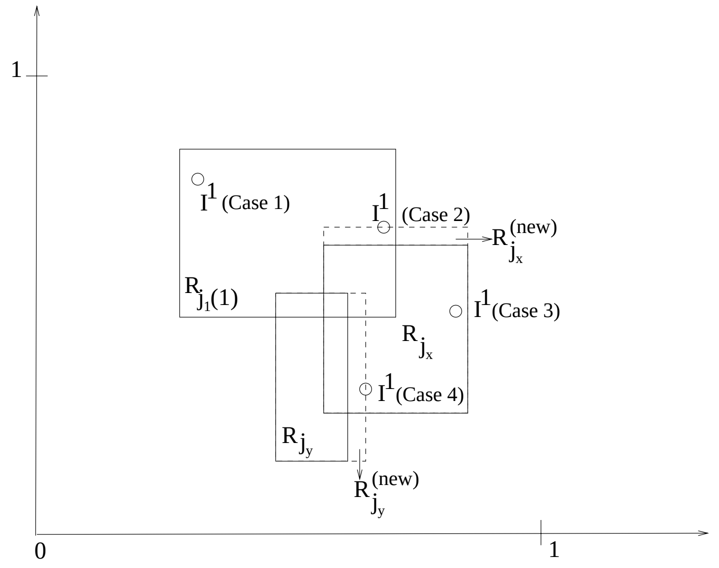


Fig. 4. Illustration that the presentation of pattern $\mathbf{I}^1$ in the second list presentation does not change the identity or size of rectangle $R_{j_1(1)}$. Case 1: $\mathbf{I}^1$ is inside $R_{j_1(1)}$ and is coded by $R_{j_1(1)}$. Case 2: $\mathbf{I}^1$ is inside $R_{j_1(1)}$ and is coded by $R_{j_x}$ ($|R_{j_x}^{\text{new}}| < |R_{j_1(1)}|$). Case 3: $\mathbf{I}^1$ is inside $R_{j_x}$ and is coded by $R_{j_x}$ ($|R_{j_x}| < |R_{j_1(1)}|$). Case 4: $\mathbf{I}^1$ is inside $R_{j_x}$ and is coded by $R_{j_y}$ ($|R_{j_y}^{\text{new}}| < |R_{j_x}| < |R_{j_1(1)}|$).

Variant training algorithm, to find the value for the parameter $N$.

**Result 3.** Consider the off-line training of a list of input patterns using the Fuzzy ART Variant algorithm. Assume that after the first list presentation the Fuzzy ART Variant algorithm has created $N$ categories in $F_2$. Then, training will be over in at most $N$ list presentations.

### 4.2. Proof of the results

The proofs of the results are based on parts 1 and 2 of Result B. Result B is proven in Georgiopoulos et al. (1996). Parts 1 and 2 of Result B were pictorially illustrated in Fig. 3 (a) and (b).

**Proof of Result 1.** Result 1 will be proven, by induction, in two steps.

- *Step* 1: Prove that Result 1 is valid for $n = 1$. That is, prove that

$$\left.\begin{array}{l} j_1(t) = j_1(1) \\ |R_{j_1(t)}| = |R_{j_1(1)}| \end{array}\right. \quad \text{for } t \geq 2. \tag{11}$$

- *Step* 2: Assume that Result 1 is valid for all indices $\leq n$ (where $n \geq 1$), and demonstrate its validity for index $n + 1$.

Hence, by assuming that

$$\begin{array}{rclc} j_1(t) & = & j_1(1) & t \geq 2 \\ |R_{j_1(t)}| & = & |R_{j_1(1)}| & t \geq 2 \\ \vdots & \vdots & \vdots & \vdots \\ j_n(t) & = & j_n(n) & t \geq n+1 \\ |R_{j_n(t)}| & = & |R_{j_n(n)}| & t \geq n+1 \end{array} \tag{12}$$

it will be proven that is true for index $n + 1$, that is

$$\left.\begin{array}{l} j_{n+1}(t) = j_{n+1}(n+1) \\ |R_{j_{n+1}(t)}| = |R_{j_{n+1}(n+1)}| \end{array}\right. \quad \text{for } t \geq n+2 \tag{13}$$

To demonstrate the validity of Step 1 two steps are needed.

- *Step 1a*: Verify Eq. (11) for list presentation $t = 2$.
- *Step 1b*: Assume the validity of Eq. (11) for list presentations 2,…, $t$ (where $t \geq 2$) and then show the validity of Eq. (11) for list presentation $t + 1$.

*Step 1a*: To illustrate Step 1a it will be shown that the identity of the node with the largest rectangle (i.e. $j_1(1)$) and the size of the largest rectangle (i.e. $|R_{j_1(1)}|$) stay intact after the presentation of the first input pattern in list presentation 2 (i.e. pattern $\mathbf{I}^1$). Then, by assuming that the identity of the node with the largest rectangle and the size of the largest rectangle stay intact after the presentation of the first $p$ input patterns in list presentation 2 (i.e. patterns $\mathbf{I}^1$, $\mathbf{I}^2$, …, $\mathbf{I}^p$), it will be proven that the identity of the node with the largest rectangle and the size of the largest rectangle stay intact after the presentation of the $(p+1)$th input pattern in list presentation 2 (i.e. pattern $\mathbf{I}^{p+1}$). Hence, by induction, it can be stated that the identity of the node with the largest rectangle (i.e. $j_1(1)$) and the size of the largest rectangle (i.e. $|R_{j_1(1)}|$) stay intact after the presentation of the $P$ training input patterns in list presentation 2, which is equivalent to saying that Eq. (11) is true for $t = 2$.

As a result, consider the presentation of the first input pattern $\mathbf{I}^1$, from the training list, during the second list presentation of the training list. We distinguish the following cases:

*Case 1*: Rectangle $R_{j_1(1)}(=R^{\text{old}}_{j_1(1)})$ is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_1(1)}$ (see Fig. 4, case 1). In this case, after $\mathbf{I}^1$'s presentation, node $j_1(1)$ is the node with the largest rectangle and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$.

*Case 2*: Rectangle $R_{j_1(1)}(=R^{\text{old}}_{j_1(1)})$ is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_x}(=R^{\text{old}}_{j_x})$, where $j_x \neq j_1(1)$ (see Fig. 4, case 2). Owing to Result B, in order for the above case to occur the following inequality must be valid:

$$|R^{\text{new}}_{j_x}| < |R_{j_1(1)}| \tag{14}$$

The above inequality guarantees that after $\mathbf{I}^1$'s presentation, node $j_1(1)$ is the node with the largest rectangle and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$.

*Case 3*: Rectangle $R_{j_x}(=R^{\text{old}}_{j_x})$, where $j_x \neq j_1(1)$, is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_x}$ (see Fig. 4, case 3). In this case, after $\mathbf{I}^1$s presentation, node $j_1(1)$ is the node with the largest rectangle and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$.

*Case 4*: Rectangle $R_{j_x}(=R^{\text{old}}_{j_x})$, where $j_x \neq j_1(1)$, is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_y}(=R^{\text{old}}_{j_y})$, where $j_y \neq j_x$(see Fig. 4, case 4). Obviously, due to Result B, $j_y$ cannot be node $j_1(1)$. Also, due to Result B, in order for the above case to occur the following inequality must be valid:

$$|R^{\text{new}}_{j_y}| < |R_{j_x}|. \tag{15}$$

However

$$|R_{j_x}| < |R_{j_1(1)}|. \tag{16}$$

The above two inequalities guarantee that after $\mathbf{I}^1$s presentation, node $j_1(1)$ is the node with the largest rectangle, and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$.

Cases 1–4 cover all the possible scenarios, and they illustrate that during the presentation of the first pattern $\mathbf{I}^1$ in list presentation 2, the identity of the node with the largest rectangle and the size of this rectangle remain intact. If it is now assumed that after $\mathbf{I}^1$'s, $\mathbf{I}^2$'s,…,$\mathbf{I}^p$'s presentations, node $j_1(1)$ is the node with the largest rectangle, and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$, it is easy to duplicate the above arguments (Cases 1–4) to illustrate that after pattern's $\mathbf{I}^{P+1}$ presentation, node $j_1(1)$ is the node with the largest rectangle and the size of the largest rectangle stays intact and equal to $|R_{j_1(1)}|$. Hence, by induction, Eq. (11) has been proven for $t = 2$.

*Step 1b*: If it is now assumed that Eq. (11) is valid for all list presentations 2,…,$t$ (where $t \geq 2$), that is

$$\begin{aligned} j_1(2) &= j_1(1) \\ |R_{j_1(2)}| &= |R_{j_1(1)}| \\ \vdots \quad & \vdots \quad \vdots \\ j_1(t) &= j_1(1) \\ |R_{j_1(t)}| &= |R_{j_1(1)}| \end{aligned} \tag{17}$$

then by duplicating the procedure, discussed in Step 1a, it can be demonstrated that

$$\begin{aligned} & j_1(t+1) = j_1(1) \\ & |R_{j_1(t+1)}| = |R_{j_1(1)}| \end{aligned} \tag{18}$$

The details are omitted due to their similarity with Step 1a. Hence, by induction, the validity of Eq. (11) has been proven.

One important byproduct of the proof of Eq. (11) is that the input patterns from the training list that chose and were coded by node $j_1(1)$ in the second list presentation will always choose and be coded by node $j_1(1)$ in subsequent list presentations (i.e. list presentations $\geq 3$). This is true because if pattern $\mathbf{I}$, from the training list, chose node $j_1(1)$ in the second list presentation it implies that

$$|R_{j_1(1)}| < |R^{\text{new}}_{j_x}| \tag{19}$$

for any $j_x \neq j_1(1)$. As the size of $R_{j_1(1)}$ remains intact after the first list presentation, while the sizes of other rectangles can increase, it is obvious that the above inequality stays valid in presentations of the input pattern $\mathbf{I}$ at subsequent lists. Thus, input pattern $\mathbf{I}$ will always choose and be coded by node $j_1(1)$ in list presentations $\geq 3$, and rectangle $R_{j_1(1)}$ will be the rectangle of the smallest size that contains $\mathbf{I}$.

*Step 2*: Assuming now the validity of Eq. (12), the truth of Eq. (13) can be demonstrated. As was the case with the proof of Eq. (11) an important byproduct of the assumption of the validity of Eq. (12) is that input patterns that chose and were coded by node $j_k(k)$ in list presentation $k+1$, will choose and be coded by node $j_k(k)$ in list presentations $\geq k+2$, where $1 \leq k \leq n$. To demonstrate the validity of Step 2 two steps are needed:

- *Step 2a*: Verify the validity of Eq. (13) for list presentation $t = n+2$.
- *Step 2b*: Assume the validity of Eq. (13) for all list presentations $\leq t$ (where $t \geq n+2$) and show the validity of Eq. (13) for list presentation $t+1$.

*Step 2a*: To illustrate Step 2a first it will be shown that the identity of the node with the $(n+1)$th largest rectangle (i.e. $j_{n+1}(n+1)$) and the size of the largest rectangle (i.e. $|R_{j_{n+1}(n+1)}|$) stay intact after the presentation of the first input pattern in the list presentation $n+2$ (i.e. pattern $\mathbf{I}^1$). Then, by assuming that the identity of the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stay intact after the presentation of the first $p$ input patterns (i.e. patterns $\mathbf{I}^1, \mathbf{I}^2, \ldots, \mathbf{I}^p$) in list presentation $n+2$, it will be proven that the identity of the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stay intact after the presentation of the $(p+1)$th input pattern (i.e. pattern $\mathbf{I}^{p+1}$) in list presentation $n+2$. Hence, by induction, it can then be stated that the identity of the node with the $(n+1)$th largest rectangle (i.e. $j_{n+1}(n+1)$) and the size of the $(n+1)$th largest rectangle (i.e. $|R_{j_{n+1}(n+1)}|$) stay intact after the presentation of the $P$ training input patterns in list presentation $n+2$, which is equivalent to saying that Eq. (13) is true for list presentation $t=n+2$.

As a result, consider the presentation of the first input pattern $\mathbf{I}^1$, from the training list, during the $(n+2)$ list presentation of the training list. The following cases are distinguished.

*Case 0*: Rectangle $R_{j_x}(=R_{j_x}^{\text{old}})$, where $j_x \in \{j_1(1), j_2(2), \ldots, j_n(n)\}$, is the smallest rectangle that contains $\mathbf{I}^1$. For example, if $j_x = j_2(2)$, this means that $R_{j_2(2)}$ was still the smallest rectangle that contained $\mathbf{I}^1$ in list presentation 3, as $R_{j_2(2)}$ did not change its size, while other rectangles have either increased their sizes or kept their sizes intact as the beginning of list presentation 3; it also means that $\mathbf{I}^1$ was chosen and coded by rectangle $R_{j_2(2)}$ in list presentation 3, otherwise $R_{j_2(2)}$ would not have been the smallest rectangle that contains $\mathbf{I}^1$ in list presentation $n+2$. Based on our previous statements (see comments immediately after Step 2), $\mathbf{I}^1$ will always choose and be coded by $R_{j_{(2)}}$ in list presentations $\geq 3$. Similar reasoning holds if $j_x$ is equal to $j_1(1)$ or $j_3(3)$, or…, $j_n(n)$. Hence, in this case, pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_x}$. Consequently, after $\mathbf{I}^1$s presentation, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$.

*Case 1*: Rectangle $R_{j_{n+1}(n+1)}(=R_{j_{n+1}(n+1)}^{\text{old}})$ is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_{n+1}(n+1)}$. In this case, after $\mathbf{I}^1$'s presentation, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$.

*Case 2*: Rectangle $R_{j_{n+1}(n+1)}(=R_{j_{n+1}(n+1)}^{\text{old}})$ is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ is chosen and coded by the rectangle $R_{j_x}(=R_{j_x}^{\text{old}})$, where $j_x \neq j_{n+1}(n+1)$. Owing to Result B, for the above case to occur the following inequality must be valid:

$$|R_{j_x}^{\text{new}}| < |R_{j_{n+1}(n+1)}|. \tag{20}$$

The above inequality guarantees that after $\mathbf{I}^1$s presentation, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$.

*Case 3*: Rectangle $R_{j_x}(=R_{j_x}^{\text{old}})$, where $j_x \neq j_k(k)$; $1 \leq k \leq n+1$, is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ chooses and is coded by rectangle $R_{j_x}$. In this case, after $\mathbf{I}^1$s presentation, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$.

*Case 4*: Rectangle $R_{j_x}(=R_{j_x}^{\text{old}})$, where $j_x \neq j_k(k)$; $1 \leq k \leq n+1$, is the rectangle of the smallest size that contains $\mathbf{I}^1$, and pattern $\mathbf{I}^1$ is chosen and coded by the rectangle $R_{j_y}(=R_{j_y}^{\text{old}})$, where $j_y \neq j_x$. Obviously, due to Result B, $j_y$ cannot be node $j_k(k)$, where $1 \leq k \leq n+1$. Also, due to Result B, for the above case to occur the following inequality must be valid:

$$|R_{j_y}^{\text{new}}| < |R_{j_x}| \tag{21}$$

However,

$$|R_{j_x}| < |R_{j_{n+1}(n+1)}|. \tag{22}$$

The above two inequalities guarantee that after $\mathbf{I}^1$s presentation, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$.

Cases 0–4 cover all the possible scenarios, and they illustrate that during the presentation of the first pattern $\mathbf{I}^1$ in list presentation $n+2$, the identity of the node with the $(n+1)$th largest rectangle and the size of this rectangle remain intact.

If it is now assumed that after $\mathbf{I}^1$'s, $\mathbf{I}^2$'s,…, $\mathbf{I}^p$'s, presentations, node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $n+1$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$, it is easy to duplicate the above arguments to illustrate that after pattern's $\mathbf{I}^{p+1}$ presentation node $j_{n+1}(n+1)$ is the node with the $(n+1)$th largest rectangle and the size of the $(n+1)$th largest rectangle stays intact and equal to $|R_{j_{n+1}(n+1)}|$. Hence, by induction, Eq. (13) has been proven for list presentation $t=n+2$.

*Step 2b*: If it is now assumed that

$$\begin{array}{ccc} j_{n+1}(n+2) & = & j_{n+1}(n+1) \\ |R_{j_{n+1}(n+2)}| & = & |R_{j_{n+1}(n+1)}| \\ \vdots & \vdots & \vdots \\ j_{n+1}(t) & = & j_{n+1}(n+1) \\ |R_{j_{n+1}(t)}| & = & |R_{j_{n+1}(n+1)}| \end{array} \tag{23}$$

then by duplicating the procedure discussed in Step 2a, it can be demonstrated that

$$\begin{array}{l} j_{n+1}(t+1) = j_{n+1}(n+1) \\ |R_{j_{n+1}(t+1)}| = |R_{j_{n+1}(n+1)}|. \end{array} \tag{24}$$

The details are omitted due to their similarity with Step 2a.

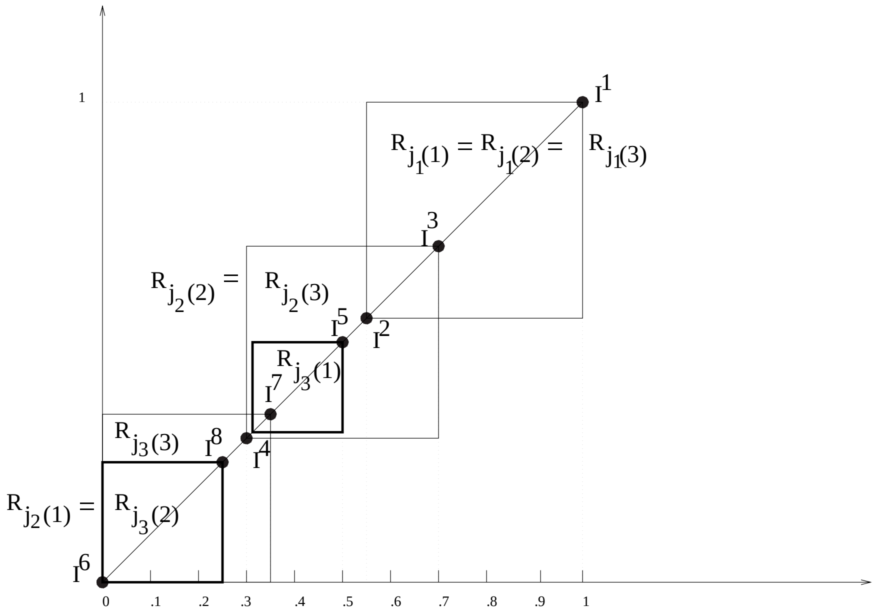


Fig. 5. Illustration of the tightness of Results 1 and 3. Rectangles created in the first three list presentations of the data $I^1 \ldots I^8$ to Fuzzy ART Variant. Data were presented from $\mathbf{I}^1$ to $\mathbf{I}^8$ in the first list presentation and from $\mathbf{I}^8$ to $\mathbf{I}^1$ in the second and third list presentation. The identity and the size of the second and the third in size rectangles change in the second list presentation (tightness of Result 1). Also, the size of the third in size rectangle changes in the third list presentation (tightness of Result 1). Finally, it takes three list presentations to learn the data (tightness of Result 5).

Hence, by induction, the validity of Eq. (13) has been proven.

**Proof of Result 2.** The above result was actually proved during the proof of Result 1 (see comments in the proof of Result 1 immediately prior, and immediately after the beginning of the proof of Step 2).

**Proof of Result 3.** The proof of this result is an immediate consequence of Result 1. This is true because the application of Result 1, at the end of list presentations $1, 2, 3, \ldots, N$ implies that the sizes of the $N$ largest rectangles do not change after list presentation $N$. If none of the $N$ rectangles change after list presentation $N$, then none of the $N$ weight vectors changes after list presentation $N$, which is equivalent to saying that *learning in the Fuzzy ART Variant* is over in at most $N$ list presentations.

## 4.3. Example

One of the interesting questions that arises, pertinent to Results 1 and 3, is the tightness of the results. This is especially important for Result 3, which gives us an upper bound on the number of list presentations required by the Fuzzy ART Variant to cluster a list of input patterns. Below, we present a simple example that illustrates the tightness of Results 1 and 3.

In this example we have eight input patterns, listed below, that are presented to the Fuzzy ART Variant in the order $\mathbf{I}^1$, $\mathbf{I}^2, \ldots, \mathbf{I}^7$, $\mathbf{I}^8$, in the first list presentation, and in the order $\mathbf{I}^8$, $\mathbf{I}^7, \ldots, \mathbf{I}^2$, $\mathbf{I}^1$ in list presentations 2 and 3. The vigilance parameter $\rho$ is chosen to equal 0.55.

$$\mathbf{I}^1 = (\mathbf{a}(1), \mathbf{a}^c(1)) = (1, 1, 0, 0) \tag{25}$$

$$\mathbf{I}^2 = (\mathbf{a}(2), \mathbf{a}^c(2)) = (0.55, 0.55, 0.45, 0.45)$$

$$\mathbf{I}^3 = (\mathbf{a}(3), \mathbf{a}^c(3)) = (0.7, 0.7, 0.3, 0.3)$$

$$\mathbf{I}^4 = (\mathbf{a}(4), \mathbf{a}^c(4)) = (0.3, 0.3, 0.7, 0.7)$$

$$\mathbf{I}^5 = (\mathbf{a}(5), \mathbf{a}^c(5)) = (0.5, 0.5, 0.5, 0.5)$$

$$\mathbf{I}^6 = (\mathbf{a}(6), \mathbf{a}^c(6)) = (0, 0, 1, 1)$$

$$\mathbf{I}^7 = (\mathbf{a}(7), \mathbf{a}^c(7)) = (0.35, 0.35, 0.65, 0.65)$$

$$\mathbf{I}^8 = (\mathbf{a}(8), \mathbf{a}^c(8)) = (0.24, 0.24, 0.76, 0.76)$$

It is not difficult to show that

1. After the first list presentation we create three categories (i.e. $N = 3$), where $j_1(1) = 1$, $j_2(1) = 3$, $j_3(1) = 2$, and $|R_{j_1(1)}| = 0.9$, $|R_{j_2(1)}| = 0.48$, $|R_{j_3(1)}| = 0.4$.
2. After the second list presentation $j_1(2) = 1$, $j_2(2) = 2$, $j_3(2) = 3$, and $|R_{j_1(2)}| = 0.9$, $|R_{j_2(2)}| = 0.8$, $|R_{j_3(2)}| = 0.48$. Hence, in the second list presentation the identity and the size of the second and third in size rectangles change (as predicted by Result 1).
3. After the third list presentation $j_1(3) = 1, j_2(3) = 2, j_3(3) = 3,$

Table 1
Comparisons of the clustering performances of Fuzzy ART (small choice parameter values $\alpha$) and Fuzzy ART Variant ($\alpha \to \infty$)

| Network parameters | | Iris database | | | | Glass database | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\rho$ | $\mathrm{Avg}_{\mathrm{cl}}$ | $\mathrm{Std}_{\mathrm{cl}}$ | $\mathrm{Avg}_{\mathrm{cat}}$ | $\mathrm{Std}_{\mathrm{cat}}$ | $\mathrm{Avg}_{\mathrm{cl}}$ | $\mathrm{Std}_{\mathrm{cl}}$ | $\mathrm{Avg}_{\mathrm{cat}}$ | $\mathrm{Std}_{\mathrm{cat}}$ |
| 0.01 | 0.4 | 74.40 | 7.27 | 5 | 0 | 44.11 | 4.20 | 6 | 0 |
| 0.1 | 0.4 | 76.33 | 8.92 | 5 | 0 | 43.73 | 3.51 | 6 | 0 |
| 1.0 | 0.4 | 77.06 | 5.16 | 4 | 1 | 43.41 | 3.81 | 6 | 0 |
| ∞ | 0.4 | 74.00 | 4.30 | 6 | 1 | 43.69 | 3.77 | 7 | 1 |
| 0.01 | 0.6 | 87.00 | 3.14 | 7 | 0 | 54.90 | 4.31 | 14 | 1 |
| 0.1 | 0.6 | 86.93 | 4.04 | 7 | 0 | 55.41 | 4.43 | 14 | 1 |
| 1.0 | 0.6 | 87.19 | 4.88 | 8 | 0 | 54.15 | 2.52 | 14 | 0 |
| ∞ | 0.6 | 86.13 | 3.57 | 8 | 1 | 51.35 | 1.98 | 16 | 1 |
| 0.01 | 0.8 | 93.40 | 2.03 | 22 | 1 | 63.22 | 2.26 | 31 | 1 |
| 0.1 | 0.8 | 93.46 | 2.06 | 22 | 1 | 63.17 | 2.36 | 31 | 1 |
| 1.0 | 0.8 | 94.13 | 2.10 | 22 | 1 | 61.49 | 3.31 | 31 | 2 |
| ∞ | 0.8 | 93.66 | 1.20 | 24 | 1 | 64.43 | 3.17 | 34 | 2 |

and $|R_{j_1(3)}| = 0.9$, $|R_{j_2(3)}| = 0.8$, $|R_{j_3(3)}| = 0.7$. Hence, in the third list presentation the size of the third in size rectangle changes (as predicted by Result 1).

4. In subsequent list presentations (i.e. list presentations $\geq 4$) the identity and size of all the rectangles remain intact, which implies that learning in the Fuzzy ART Variant is over in three list presentations (as predicted by Result 3).

The rectangles created in this example after list presentations 1, 2, and 3 are shown in Fig. 5. In Fig. 5, we see in a pictorial fashion the tightness of Results 1 and 3.

## 5. Simulations

In order to assess the practicality of the Fuzzy ART Variant (Fuzzy ART with large values of the choice parameter) compared to typical Fuzzy ART (Fuzzy ART with small values of the choice parameter) we evaluated the performance of these algorithms on two databases. These databases were chosen from the collection of databases found at the UCI repository (Murphy & Aha, 1994). The databases chosen were: Iris and Glass. The Iris database is perhaps the best known database to be found in the pattern recognition literature. The data set consists of three classes of 50 instances each, where each class refers to an iris plant (*Iris setosa, Iris versicolour* and *Iris virginica*). The number of features for each instance are the sepal length, the sepal width, the petal length, and the petal width, all in centimeters. For the training of Fuzzy ART (Fuzzy ART Variant) the $P = 150$ input patterns are used, where each input pattern has $M = 4$ components. The glass database is used for classification of some type of glass. This database was motivated by criminological investigation, where the glass left at the crime scene, can be used as evidence. Each instance has nine features and it can be classified as one of six classes. There are 214 instances of input–output pairs. For the training of Fuzzy ART (Fuzzy ART Variant) the $P = 214$ input patterns are used, where each input pattern has $M = 9$ components.

To evaluate the clustering performance of Fuzzy ART (Fuzzy ART Variant) we trained it with the training list of input patterns until it learned the list completely. After training was over we assigned a label to each category formed in the $F_2$ field. A category formed in the $F_2$ field takes the label of the output pattern to which most of the input patterns that chose this category belong. For example, consider the case where we have ten input patterns, named $\mathbf{I}^1, \mathbf{I}^2, \ldots, \mathbf{I}^{10}$, in the training list. Assume that the label of patterns $\mathbf{I}^1$ through $\mathbf{I}^4$ is $\mathbf{O}^1$, and the label of patterns $\mathbf{I}^5$ through $\mathbf{I}^{10}$ is $\mathbf{O}^2$. Assume also that three nodes in $F_2$ were committed during the training process of Fuzzy ART; these nodes are named nodes 1, 2, and 3. After training is over, we present the input patterns in the training collection one more time through Fuzzy ART. Suppose that patterns $\mathbf{I}^1$, $\mathbf{I}^2$, $\mathbf{I}^3$ and $\mathbf{I}^5$ choose node 1 in $F_2$, patterns $\mathbf{I}^4$, $\mathbf{I}^6$ and $\mathbf{I}^7$ choose node 2 in $F_2$, and finally patterns $\mathbf{I}^8$, $\mathbf{I}^9$ and $\mathbf{I}^{10}$ choose node 3 in $F_2$. As three out of the four input patterns that choose node 1 need to be mapped to output pattern $\mathbf{O}^1$, the label for node 1 is output pattern $\mathbf{O}^1$. Also, as two of the three nodes that choose node 2 need to be mapped to pattern $\mathbf{O}^2$, the label for node 2 is output pattern $\mathbf{O}^2$. Finally, as three out of the three input patterns that choose node 3 need to be mapped to output node $\mathbf{O}^2$, the label of node 3 is also output pattern $\mathbf{O}^2$. We see from above that two patterns from the input training collection go to a node whose label is different than the output pattern to which the input pattern needs to be mapped to. For example, pattern $\mathbf{I}^5$ needs to be mapped to output pattern $\mathbf{O}^2$ but chooses node 1 with label $\mathbf{O}^1$, and pattern $\mathbf{I}^4$ needs to be mapped to output pattern $\mathbf{O}^1$ but chooses node 2 with label $\mathbf{O}^2$. We say that patterns $\mathbf{I}^4$ and $\mathbf{I}^5$ are incorrectly clustered, while the rest of the input patterns (i.e. patterns $\mathbf{I}^1$ through $\mathbf{I}^3$, and $\mathbf{I}^6$ through $\mathbf{I}^{10}$) are correctly classified. As a result, the clustering performance of Fuzzy ART in the above example is 80%. Similarly, we can evaluate the clustering performance of Fuzzy ART Variant. The aforementioned procedure to evaluate the clustering performance of clustering algorithms was initially introduced by Dubes and Jain (1976).

The average clustering performance ($Avg_{cl}$) of Fuzzy ART (Fuzzy ART Variant) is depicted in Table 1. The average clustering performance of Fuzzy ART (Fuzzy ART Variant) is calculated by evaluating the average of the clustering performances of ten networks trained by Fuzzy ART (Fuzzy ART Variant) for different orders of training pattern presentations. In Table 1, other measures of performance are depicted such as standard deviation of the clustering performances ($Std_{cl}$), as well as average number of categories formed in $F_2$ ($Avg_{cat}$), and standard deviation of the number of categories formed in $F_2$ ($Std_{cat}$). Our conclusion by comparing the results in Table 1 (i.e. Fuzzy ART performance for small $\alpha$ values, and Fuzzy ART performance for large $\alpha$ values (Fuzzy ART Variant)) is that Fuzzy ART Variant performance is close to that of Fuzzy ART that uses small $\alpha$ values. Similar conclusions were drawn in (Georgiopoulos et al., 1996) where the Fuzzy ART and the Fuzzy ART Variant clustering performances were compared on additional databases (e.g. Heart, Diabetes, Wine, Ionosphere, and Sonar) from the UCI repository (Murphy & Aha, 1994).

## 6. Summary and discussion

In this paper attention was focused on a Fuzzy ART Variant that is obtained if in Fuzzy ART we use very large values for the choice parameter $\alpha$, and the initial components of the weights. Useful properties of learning were developed and proved for this Fuzzy ART Variant. One of these properties of learning provided an upper bound on the number of list presentations required by this Fuzzy ART Variant to learn an arbitrary list of analog input patterns. This upper bound verified one of the important properties of this Fuzzy ART Variant, the short-training time property. Knowing that properties of learning for Fuzzy ART with small values for the choice parameters (including the short-training time property) have already been demonstrated in the literature, this paper fills an important theoretical gap. What remains to be seen is whether the short training time property is valid in Fuzzy ART for intermediate values of the choice parameter (i.e. not too small and not too large values). We have also illustrated through simulations that this Fuzzy ART Variant exhibits a clustering performance that is comparable to the Fuzzy ART (with small choice parameter values) clustering performance. Hence, it can be used in practice instead of, or in conjunction with Fuzzy ART.

## Acknowledgements

This research was supported in part by a grant from Boeing Computer Services under contract W-300445.

## References

Carpenter, G.A., & Gjaja, M.N. (1994) Fuzzy ART choice functions. *Proceedings of the World Congress on Neural Networks* (pp. I-713–722), San Diego, CA.

Carpenter, G. A., & Grossberg, S. (1987a). ART 2: Self-organization of stable category recognition codes for analog input patterns. *Applied Optics*, *26* (23), 4919–4930.

Carpenter, G. A., & Grossberg, S. (1987b). A massively parallel architecture for a self-organizing neural pattern recognition machine. *Computer Vision, Graphics and Image Processing*, *37*, 54–115.

Carpenter, G. A., & Grossberg, S. (1990). ART 3: Hierarchical search using chemical transmitters in self-organizing pattern recognition architectures. *Neural Networks*, *3* (2), 129–152.

Carpenter, G. A., Grossberg, S., Markuzon, N., Reynolds, J. H., & Rosen, D. B. (1992). Fuzzy ARTMAP: A neural network architecture for incremental supervised learning of analog multidimensional maps. *IEEE Transactions on Neural Networks*, *3* (5), 698–713.

Carpenter, G. A., Grossberg, S., & Reynolds, J. H. (1991a). ARTMAP: Supervised real-time learning and classification of nonstationary data by a self-organizing neural network. *Neural Networks*, *4* (5), 565–588.

Carpenter, G. A., Grossberg, S., & Rosen, D. B. (1991b). Fuzzy ART: Fast stable learning and categorization of analog patterns by an adaptive resonance system. *Neural Networks*, *4* (6), 759–771.

Carpenter, G. A., & Markuzon, N. (1998). ARTMAP-IC and medical diagnosis: Instance counting and inconsistent cases. *Neural Networks*, *11*, 323–326.

Carpenter, G. A., & Ross, W. D. (1995). ART-EMAP: A neural architecture for object recognition by evidence accumulation. *IEEE Transactions on Neural Networks*, *6*, 805–818.

Dubes, R., & Jain, A. (1976). Clustering techniques: The user's dilemma. *Pattern Recognition*, *8*, 247–260.

Georgiopoulos, M., Fernlund, H., Bebis, G., & Heileman, G. L. (1996). Order of search in Fuzzy ART and Fuzzy ARTMAP: Effect of the choice parameter. *Neural Networks*, *9* (5), 1541–1559.

Grossberg, S. (1976). Adaptive pattern recognition and universal recoding II. Feedback, expectation, olfaction, and illusions. *Biological Cybernetics*, *23*, 187–202.

Healy, M. J., Caudell, T. P., & Smith, S. D. G. (1993). A neural architecture for pattern sequence verification through inferencing. *IEEE Transactions on Neural Networks*, *4* (1), 9–20.

Huang, J., Georgiopoulos, M., & Heileman, G. L. (1995). Fuzzy ART properties. *Neural Networks*, *8* (2), 203–213.

Marriott, S., & Harrison, R. (1995). A modified Fuzzy ARTMAP architecture for the approximation of noisy mappings. *Neural Networks*, *8* (4), 619–641.

Murphy, P., & Aha, D. (1994). UCI Repository of Machine Learning Databases. Technical report, Department of Computer Science [www.ics.edu/mlearn/MLRepository.html]. University of California, Irvine, CA.

Tan, A. H. (1995). Adaptive resonance associative map. *Neural Networks, 8* (3), 437–446.

Williamson, J. R. (1996). Gaussian ARTMAP: A neural network for fast incremental learning of noisy multidimensional maps. *Neural Networks*, *9* (5), 881–897.