Shape Metamorphism using p-Laplacian Equation

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ABSTRACT

We present a new approach for shape metamorphism, which is a process of gradually changing a source shape (known) through intermediate shapes (unknown) into a target shape (known). The problem, when represented with implicit scalar function, is under-constrained, and regularization is needed. Using the p-Laplacian equation (PLE), we generalize a series of regularization terms based on the gradient of the implicit function, and we show that the present methods lack additional constraints for a more stable solution. The novelty of our approach is in the deployment of a new regularization term when $p \to \infty$ which leads to the infinite Laplacian equation (ILE). We show that ILE minimizes the supremum of the gradient and prove that it is optimal for metamorphism since intermediate solutions are equally distributed along their normal direction. Applications of the proposed algorithm for 2D and 3D objects are demonstrated.

1 Introduction

Application of shape metamorphosis has the potential of going beyond simple animation and interpolation. Ultimately, one can envision using metrics computed during shape interpolation for comparative analysis and matching. Furthermore, if morphing can be performed efficiently, then the gradual transition between initial and final shape can be subsampled for distributed rendering. Metamorphosis is often referred to as the process of smooth topological transition between two objects. Metamorphosis is different from interpolating function values at different grid points because new shapes are reconstructed from two known shapes, and it is different from the shape evolution problem because initial and final solutions are known.

Lazarus and Verroust [8] provide an excellent survey and evaluation of 3D morphing techniques. Some of these techniques are initiated from the association of vertices and triangles between source and target meshes. Other methods use implicit functions obtained by linear interpolation of shape transformation of three-dimensional images. Our method is more aligned with the application of evolution equation [3] or distance field manipulation (DFM) [3, 8]. In DFM, the distance transformations from initial to final objects are represented as regular functions, while intermediate implicit functions are approximated by linear interpolation between corresponding distance transformations. Each intermediate shape is then extracted as the zero-crossing of the corresponding intermediate function [9, 10, 11, 13]. Although DFM has been widely used, it has not been sufficiently questioned. For example: (1) Why does the distance transformation work? (2) Are there alternative methods? (3) If there is an alternative method, then is the distance transformation the optimal field function? This paper partially focuses on these questions by exploring an entirely new approach.

Here we propose a new approach for metamorphosis of shapes. With the p-Laplacian equation (PLE), we generalize a series of regularized terms based on the gradient of the implicit function. We show that present methods are subsets of our formulation that limit the solution to the metamorphosis problem. The novelty of our approach is in the deployment of a new regularization term when $p \to \infty$, which leads to the infinite Laplacian equation (ILE) that minimizes the supremum of the gradient. This approach is *optimal* for the metamorphosis problem since the shapes are equally distributed along the gradient trajectory. We also show that while DFM is efficient and simple, it is only an approximate solution to the ILE and can be used as the initial value to solve the equation. We also show that the metamorphosis technique can be used for multiscale shape representation.

Section 2 outlines two-dimensional representation and the proposed new regularization framework. Section 3 outlines the corresponding numerical solution and compares the behavior of the energy function to the standard gradient-based regularization. In Section 4, we extend the method to three dimensions. Section 5 provides experimental results and the application of the proposed methods to multi-scale shape representation. Section 6 concludes the paper.

*This work is supported in part by the Director, Office of Energy Science Research, Office of Computation and Technology Research, Mathematical, Information, and Computational Sciences of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098 with the University of California, and the National Sciences Foundation CRCD grant No.0088086. Publication ID is LBNL-55031.

2 The p-Laplacian Equation and Regularization Terms

In this section, we study the PLE and derive different regularization terms from this equation. We then establish the relationships among curve evolution, energy minimization, regularization terms, PLE, and ILE. Our study first focuses on two-dimensional curves, and is then extended to three-dimensional surfaces.

2.1 The Problem

Let $\mathcal{S}$ be a deformable closed curve such that $\mathcal{S}(t), t \in [0,1]$ denotes a family of the evolved curves with known boundary condition at $\mathcal{S}(0) = \mathcal{S}_0$ and $\mathcal{S}(1) = \mathcal{S}_1$. Our aim is to reconstruct a representation between $\mathcal{S}(t), 0 < t < 1$ so that the sequence of intermediate curves is smooth and continuous in time. Let $\mathcal{O}_i$ be the inside-outside function of a closed curve $\mathcal{S}_i$, $i = 0, 1$ such that

$$\mathcal{O}_i(x,y) = \begin{cases} -1, & \text{if } (x,y) \text{ is inside } \mathcal{S}_i \\ 1, & \text{if } (x,y) \text{ is outside } \mathcal{S}_i \\ 0, & \text{if } (x,y) \text{ is on } \mathcal{S}_i \end{cases} \quad (1)$$

We then define the "Metamorphosis Region" $\mathcal{R}$ as

$$\mathcal{R}(\mathcal{S}_0, \mathcal{S}_1) = \{(x,y) | \mathcal{O}_0(x,y)\mathcal{O}_1(x,y) \le 0\} \quad (2)$$

Examples of $\mathcal{R}$ in two dimensions are shown in Figure 1. We restrict the metamorphosis to $\mathcal{R}$, *i.e.*, $\mathcal{S}(t) \subset R$. As t changes from 0 to 1, $\mathcal{S}(t)$ changes from $\mathcal{S}_0$ to $\mathcal{S}_1$ continuously and smoothly, and sweeps every point in $\mathcal{R}$. Since the metamorphosis is time dependent and constrained under the initial and final surfaces, these intermediate curves can be expressed as the level curves of an implicit function. Then the problem becomes to find a function $f(x,y)$, defined in $\mathcal{R}$, such that $\mathcal{S}(t)$ is determined by $f(x,y) = t$. Formally, suppose $f(x,y)$ is the time at which the curve crosses a given point (x,y). Then function f satisfies

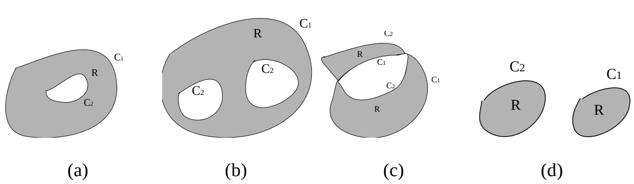


Figure 1: $\mathcal{R}(C_0, C_1)$: (a) doubly connected region, (b) multiply connected region, (c) intersection, and (d) isolated objects.

$$\mathcal{S}(t) = \{(x,y) | f(x,y) = t\} \quad (3)$$

Equation (3) gives an implicit representation of the curves. Thus, our problem is:

Find $f(x,y), (x,y) \in \mathcal{R}$, such that $f(\mathcal{S}_0) = 0, f(\mathcal{S}_1) = 1$.

2.2 A New Regularization Term

The problem of reconstructing $f(x,y)$ is certainly underconstrained. However, since we changed the problem from curve metamorphosis to functional interpolation, well known mathematical tools, such as regularization, can be leveraged. A majority of the existing regularization techniques attempt to minimize an integral such as $\int_R |\bigtriangledown f|$ or $\int_R |\bigtriangledown f|^2$ [1, 4, 12]. The idea behind these approaches is to minimize the global variation of f. Yet, this formulation has no control on the local property of f. In other words, the global variation may be small, but locally f may change sharply. Consider the following energy function:

$$\epsilon = \int_R |\bigtriangledown f|^n dxdy \quad (4)$$

Present regularized methods assume either $n = 1, 2$, which provides little local control; however, our approach generalizes them. The corresponding Euler equation of (4) is given by:

$$div(|\bigtriangledown f|^{n-2} \bigtriangledown f) = 0 \quad (5)$$

where div is the diverge operator. Equation (5) is called the p-Laplacian equation (or p-harmonic function in some literature). When $n \to \infty$, we have

$$\bigtriangledown(|\bigtriangledown f|) \cdot \frac{\bigtriangledown f}{|\bigtriangledown f|} = 0 \quad (6)$$

It was shown that when $n \to \infty$, we are actually minimizing the super norm of $|\bigtriangledown f|$ [7]. Rewriting equation (6), we have:

$$f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy} = 0 \quad (7)$$

2.3 Equal Importance Criteria

This section outlines the rationale for optimality of the supremum as the norm for regularization. Our argument is based on the *equal importance criterion* [5, 6]. This criterion asserts that *every point in $\mathcal{R}$ is equally important and contributes similarly to the reconstruction process. Any other assumption means that we need to know some additional information about the curve.* Equation (6) implies that along each trajectory of the gradient of f, the magnitude of the gradient is a constant. The interpolated curves $\mathcal{S}(t)$ are then equally distributed along their normal direction, or simply each point advances at its own constant speed. In the absence of any information about the deformation process, once can only assume that the curves $\mathcal{S}(t)$ are equally distributed. Thus, in view of time or distance between curves, which is our only clue about the curve, all points are equally important. Comparative analysis of our approach indicates that our method generates a more smooth family of curves. The aim is to minimize supremum of the maximum. Although the overall integral of $|\bigtriangledown f|$ may be larger, the supremum is smaller in our approach and the gradient is more likely to concentrate at a smaller range.

Thus, the speed of the moving curve is in a smaller range and the whole curve changes more smoothly with time.

Introducing the notation $\mathcal{J}(f) = f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy}$, our problem now becomes

Find $f(x, y), (x, y) \in \mathcal{R}$, *such that*
$f(\mathcal{S}_0) = 0, f(\mathcal{S}_1) = 1, and\ \mathcal{J}(f) = 0$

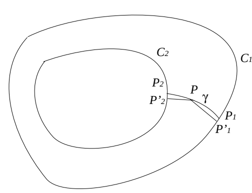
3 Numerical Solution of ILE

Many numerical methods can be used to solve Equation (7), and at least a weak solution is guaranteed. Our approach uses a variation of gradient decent with a good initial condition for efficient convergence.

3.1 Initialization with the Distance Transform

Let us define $\mathcal{T}_i$ as the *signed distance transformation* of $\mathcal{S}_i, i = 0, 1$, where $\mathcal{T}_i(x, y)$ is the distance from (x, y) to the nearest point on $\mathcal{S}_i$, and the distance is set to a negative number if (x, y) is inside $\mathcal{S}_i$ and positive otherwise. For each point p (shown in Figure 2), there should be a gradient trajectory γ passing through it such that it intersects $\mathcal{S}_0$ and $\mathcal{S}_1$ at p_0 and p_1, respectively. Since the normal of these two curves and the gradient of f are in the same direction, $\gamma \perp \mathcal{S}_0$ at p_0 and $\gamma \perp \mathcal{S}_1$ at p_1, where $\perp$ represents perpendicular. We can approximate the curve γ passing through p, by drawing two line segments $pp_0' \perp \mathcal{S}_0, pp_1' \perp \mathcal{S}_1$, to create $p_0'pp_1'$. Let l denote the length of γ from p_0 to p_1. Hence, $l \approx |p_0'p| + |p_1'p|$. The preceding formulation indicates that $|p_0'p| = -\mathcal{T}_0(p)$, $|p_1'p| = \mathcal{T}_1(p)$. Since f changes linearly from 0 to 1 along γ, $f(p)$ can be approximated by:

$$f(p) = \frac{|p_0'p|}{|p_0'p| + |p_1'p|} = \frac{-\mathcal{T}_0(p)}{\mathcal{T}_1(p) - \mathcal{T}_0(p)} \tag{8}$$



(a)

Figure 2: Computing $f(x, y)$ from distance transformation.

Equation (8) has a drawback that, when $\mathcal{S}_0$ and $\mathcal{S}_1$ intersect each other at p, we get $\mathcal{T}_1(p) - \mathcal{T}_0(p) = 0$ and a zero divisor. Alternatively, the two-dimensional isocurve representation, $\phi(x, y, t), 0 \leq t \leq 1$, can be expressed as:

$$\phi(x, y, t) = t\mathcal{T}_1(x, y) + (1 - t)\mathcal{T}_0(x, y) \tag{9}$$

Note that this is exactly the DFM, and isocurve $\phi(x, y, t) = 0$ is located at

$$t(x, y) = \frac{-\mathcal{T}_0(x, y)}{\mathcal{T}_1(x, y) - \mathcal{T}_0(x, y)} \tag{10}$$

which is exactly the curve that we reconstructed in (8). Thus, DFM is an approximation of the ILE solution in the sense described above. Equation (9) is preferred over equation (8) because it works for any $\mathcal{S}_0$ and $\mathcal{S}_1$ even if $\mathcal{S}_0 = \mathcal{S}_1$. Thus, the method treats any curve and topological changes naturally and cannot fail. Let $0 \leq t \leq 1$ in equation (10) then $\mathcal{T}_1(x, y)\mathcal{T}_0(x, y) \leq 0$. We know that $\mathcal{T}_1(x, y)\mathcal{T}_0(x, y) \leq 0$ in and only in $\mathcal{R}$. Thus, the isocurve $\phi(x, y, t) = 0$ is guaranteed to be inside the region R, $\mathcal{S}(t) \subset R, 0 \leq t \leq 1$.

3.2 The Iterative Approach

The solution of ILE can be obtained iteratively by:

$$f^{v+1} = f^v - \delta \mathcal{J}(f^v) \tag{11}$$

where δ is the step size and v indicates the iteration number. Although the approach converges only to a local minimum, the solution is acceptable if we start from a good initialization. The whole algorithm can be summarized as:
1) Initialize f, with the boundary condition $f(\mathcal{S}_0) = 0, f(\mathcal{S}_1) = 1$.
2) Initialize $\mathcal{R}$ with the distance transform.
3) Update all the points inside R with equation (11).
4) Compute $sup| \bigtriangledown f|$.
5) Repeat 2 to 3 until a local minimum of $sup| \bigtriangledown f|$ is reached, and
6) Find the interpolated curves $\mathcal{S}(t) = \{(x, y)|f(x, y) = t\}$.

4 Extension to Three-Dimensional Surfaces

The extension to three-dimensional objects is through rewriting the ILE as:

$$f_x^2 f_{xx} + f_y^2 f_{yy} + f_x^2 f_{zz} + 2 f_x f_y f_{xy} + 2 f_y f_z f_{yz} + 2 f_y f_x f_{zx} = 0 \tag{12}$$

An exact representation of the distance transform in 3D is compute-intensive; however, a quick approximation [2] is sufficient for smooth interpolation. The distance transform is computed in two passes over the bounding volume where the volume is represented as a dense 3D array, whose values are initialized with initial and final surfaces.

5 Applications and Experimental Results

In this section, experimental results corresponding to the application of proposed techniques to 2D and 3D objects are shown. In every experiment, the first curve is the source curve and the last scurve is the target curve, while others are interpolated intermediate curves. Figures 3 show the results computed by the variational approach where f is initialized by DFM. We then update the implicit function until a local minimum is reached. Finally, the level curves at different heights are extracted. Figure 3 shows interpolation of a fish shape to a panda. The image size is 162 by 161 where 8 curves have been inserted. In this experiment, $\delta = 0.05$.

In this experiment, $\delta = 0.03$ and the regularization step is chosen to be a small value of 0.001. The distance transformation is computed by the methods proposed by Borgefors [2]. Figure 4 shows eight snapshots corresponding to transformation of a diseased cortex to a normal one. In this case, corresponding MRI images were segmented and the white matter in the cortex was automatically delineated. Figure 5 shows another eight snapshots corresponding to the transformation of a Buddha to a bunny.

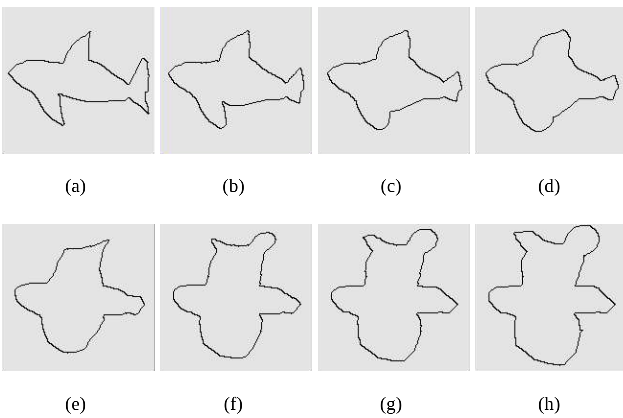


Figure 3: Interpolating fish and panda.

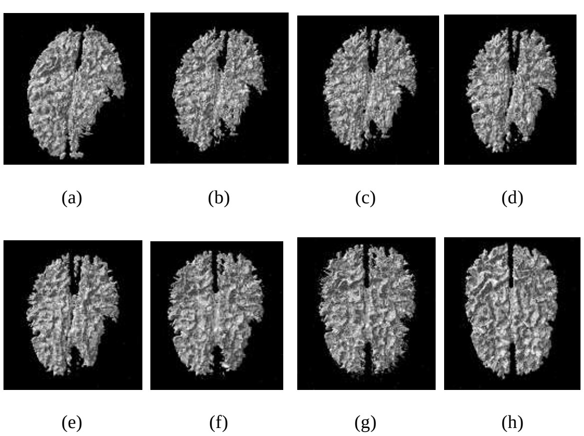


Figure 4: Morphing from a diseased brain to a normal one

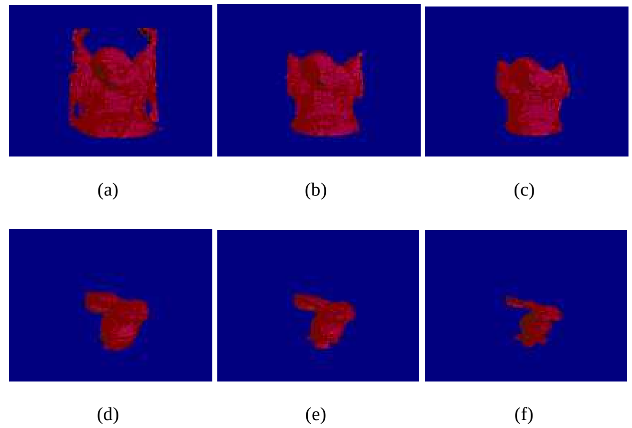


Figure 5: Morphing from Buddha to a bunny.

6 Conclusion

We proposed a p-Laplacian-based solution to the shape morphing problem. Our method finds natural and smooth shapes that are equally distributed along the normal direction. This is optimal when no information about the deformation process exists, and the best thing we can do is to assign the generated shapes fairly. The PDE is derived from a new regularization term that ensures the local smoothness. A numerical method was developed to construct and compute an optimal solution. At the same time, we showed that DFM is an efficient and simple approximation to the ILE, which can handle any curve with arbitrary topological changes.

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