

CS 474/674 Image Processing & Interp.  
Fall 2009 - HW2

Solutions

$$1) \mathcal{F}(e^{-\pi x^2}) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi u x} dx =$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + j2ux)} dx =$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + j2ux + (ju)^2 - (ju)^2)} dx =$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x+ju)^2} e^{-\pi u^2} dx = e^{-\pi u^2} \int_{-\infty}^{\infty} e^{-\pi(x+ju)^2} dx$$

(set  $x+ju=t \Rightarrow dx=dt$ )

$$= e^{-\pi u^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi t^2} dt}_1 = e^{-\pi u^2}$$

### Problem 4.18

We consider the 1-D case first. From Eq. (4.4-6),

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

When  $f(x) = 1$  and  $u = 0$ ,  $F(u) = 1$ . When  $f(x) = 1$  and  $u \neq 0$ ,

$$F(u) = \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \quad [1, M-1]$$

This expression is 0 for any integer value of  $u$  in the range  $[0, M-1]$ . There are various ways of proving this. One of the most intuitive is based on using Euler's formula to express the exponential term as

$$e^{-j2\pi ux/M} = \cos(2\pi ux/M) - j \sin(2\pi ux/M)$$

This expression describes a unit vector in the complex plane. The vector is centered at the origin and its direction depends on the value of the argument. For any integer value of  $u$  in the range  $[0, M-1]$  the argument ranges over integer values of  $x$  in the same range. This means that the vector makes an integer number of revolutions about the origin in equal increments. Thus, for any positive value of  $\cos(2\pi ux/M)$ , there will be a corresponding negative value of this term. This produces a zero sum for the real part of the exponent. Similar comments apply the imaginary part. Therefore, when  $f(x) = 1$  and  $u \neq 0$ , it follows that  $F(u) = 0$ . Thus, we have shown that for discrete quantities,

$$\mathfrak{F}[1] = \delta(u) = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{if } u \neq 0. \end{cases}$$

A similar procedure applies in the case of two variables, and we have that

$$\mathfrak{F}[1] = \delta(u, v) = \begin{cases} 1 & \text{if } u = 0 \text{ and } v = 0 \\ 0 & \text{otherwise.} \end{cases}$$

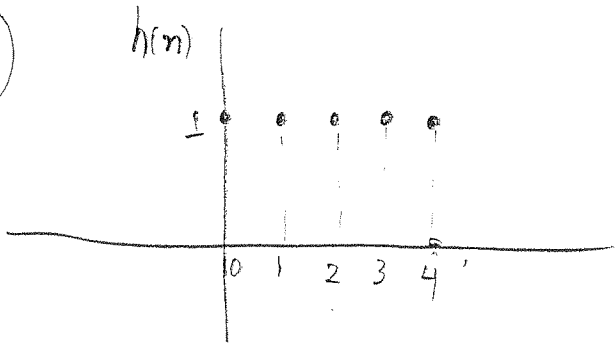
### Problem 4.19

From Eq. (4.5-15), and using the exponential representation of the sine function, we have

$$\begin{aligned} F(u, v) &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sin(2\pi u_0 x + 2\pi v_0 y) e^{-j2\pi(ux/M + vy/N)} \\ &= \frac{-j}{2} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [e^{j2\pi(u_0 x + v_0 y)} - e^{-j2\pi(u_0 x + v_0 y)}] e^{-j2\pi(ux/M + vy/N)} \\ &= \frac{-j}{2} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{j2\pi(Mu_0 x/M + Nv_0 y/N)} e^{-j2\pi(ux/M + vy/N)} \\ &\quad - \frac{-j}{2} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{-j2\pi(Mu_0 x/M + Nv_0 y/N)} e^{-j2\pi(ux/M + vy/N)} \\ &= \frac{-j}{2} \mathfrak{F}[(1)e^{j2\pi(Mu_0 x/M + Nv_0 y/N)}] + \frac{j}{2} \mathfrak{F}[(1)e^{-j2\pi(u_0 x + v_0 y)}] \\ &= \frac{j}{2} [\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)] \end{aligned}$$

where the fourth step follows from the discussion in Problem 4.17, and the last line follows from Table 4.3.

4)



$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-\frac{j2\pi ux}{N}}, \quad u=0, 1, \dots, 4$$

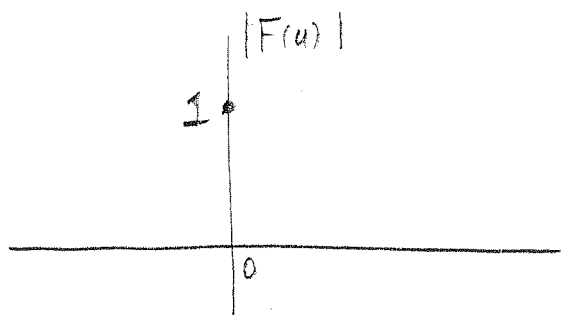
$$\underline{u=0} \quad F(0) = \frac{1}{5} [f(0)e^0 + f(1)e^0 + f(2)e^0 + f(3)e^0 + f(4)e^0] = \frac{1}{5} [1+1+1+1+1] = 1$$

$$\begin{aligned} \underline{u=1} \quad F(1) &= \frac{1}{5} \left[ f(0)e^0 + f(1)e^{-\frac{j2\pi \cdot 1}{5}} + f(2)e^{-\frac{j2\pi \cdot 2}{5}} + f(3)e^{-\frac{j2\pi \cdot 3}{5}} + f(4)e^{-\frac{j2\pi \cdot 4}{5}} \right] \\ &= \frac{1}{5} \left[ 1 + e^{-\frac{j2\pi}{5}} + e^{-\frac{j4\pi}{5}} + e^{-\frac{j6\pi}{5}} + e^{-\frac{j8\pi}{5}} \right] = \\ &= \frac{1}{5} \left[ 1 + (\cos \frac{2\pi}{5} - j \sin \frac{2\pi}{5}) + (\cos \frac{4\pi}{5} - j \sin \frac{4\pi}{5}) + (\cos \frac{6\pi}{5} - j \sin \frac{6\pi}{5}) + \right. \\ &\quad \left. + (\cos \frac{8\pi}{5} - j \sin \frac{8\pi}{5}) \right] = 0 \end{aligned}$$

$$\begin{aligned} \underline{u=2} \quad F(2) &= \frac{1}{5} \left[ f(0)e^0 + f(1)e^{-\frac{j2\pi \cdot 2 \cdot 1}{5}} + f(2)e^{-\frac{j2\pi \cdot 2 \cdot 2}{5}} + f(3)e^{-\frac{j2\pi \cdot 2 \cdot 3}{5}} + \right. \\ &\quad \left. + f(4)e^{-\frac{j2\pi \cdot 2 \cdot 4}{5}} \right] = \frac{1}{5} \left[ 1 + e^{-\frac{j4\pi}{5}} + e^{-\frac{j8\pi}{5}} + e^{-\frac{j12\pi}{5}} + e^{-\frac{j16\pi}{5}} \right] = 0 \end{aligned}$$

$$\underline{u=3} \quad F(3) = \frac{1}{5} \left[ 1 + e^{-\frac{j6\pi}{5}} + e^{-\frac{j12\pi}{5}} + e^{-\frac{j18\pi}{5}} + e^{-\frac{j24\pi}{5}} \right] = 0$$

$$\underline{u=4} \quad F(4) = \frac{1}{5} \left[ 1 + e^{-\frac{j8\pi}{5}} + e^{-\frac{j16\pi}{5}} + e^{-\frac{j24\pi}{5}} + e^{-\frac{j32\pi}{5}} \right] = 0 \quad 3$$



(Just an impulse!!)

(Grad Students Only)

5) a)  $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$  (1)

also,  $f(-x) = f_{\text{even}}(-x) + f_{\text{odd}}(-x) = f_{\text{even}}(x) - f_{\text{odd}}(x)$  (2)

(1) + (2)  $f(x) + f(-x) = 2 f_{\text{even}}(x) \Rightarrow f_{\text{even}} = \frac{1}{2} [f(x) + f(-x)]$

(1) - (2)  $f(x) - f(-x) = 2 f_{\text{odd}}(x) \Rightarrow f_{\text{odd}} = \frac{1}{2} [f(x) - f(-x)]$

b) find  $\mathcal{F}(f(-x))$  first

$$\begin{aligned} \mathcal{F}(f(-x)) &= \int_{-\infty}^{\infty} f(-x) e^{-j2\pi ux} dx = \left\| \begin{array}{l} \text{Set } t = -x \Rightarrow dt = -dx \\ \text{or } dx = -dt \end{array} \right. \\ &= - \int_{\infty}^{-\infty} f(t) e^{j2\pi ut} dt = \int_{-\infty}^{\infty} f(t) e^{j2\pi ut} dt = F^*(u) \end{aligned}$$

thus,  $\mathcal{F}(f_{\text{even}}) = \frac{1}{2} [\mathcal{F}(f(x)) + \mathcal{F}(f(-x))] = \frac{1}{2} [F(u) + F^*(u)] = \text{Re}(F(u))$

$$\mathcal{F}(f_{\text{odd}}) = \frac{1}{2} [\mathcal{F}(f(x)) - \mathcal{F}(f(-x))] = \frac{1}{2} [F(u) - F^*(u)] = j \text{Im}(F(u))$$