Epipolar (Stereo) Geometry

• Epipoles, epipolar plane, and epipolar lines

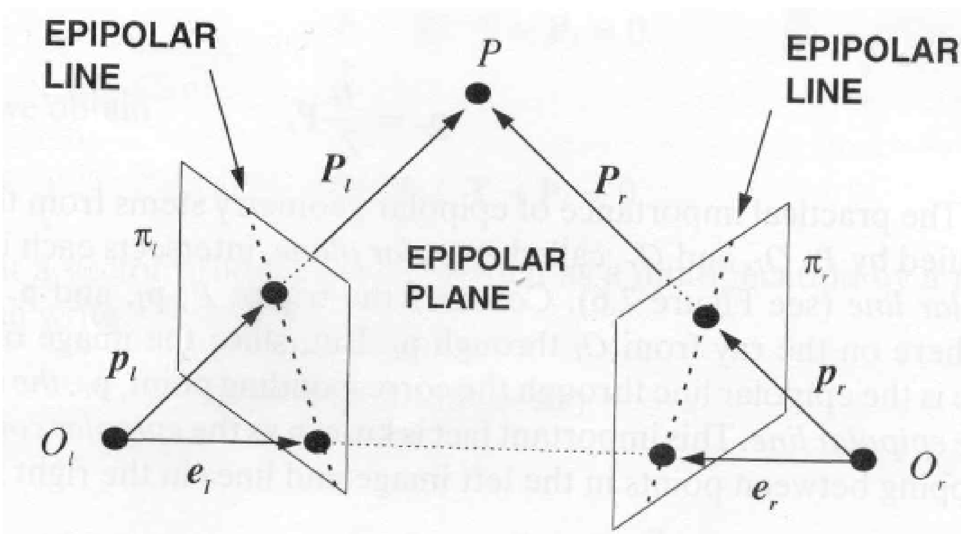
- The image in one camera of the projection center of the other camera is called *epipole*.

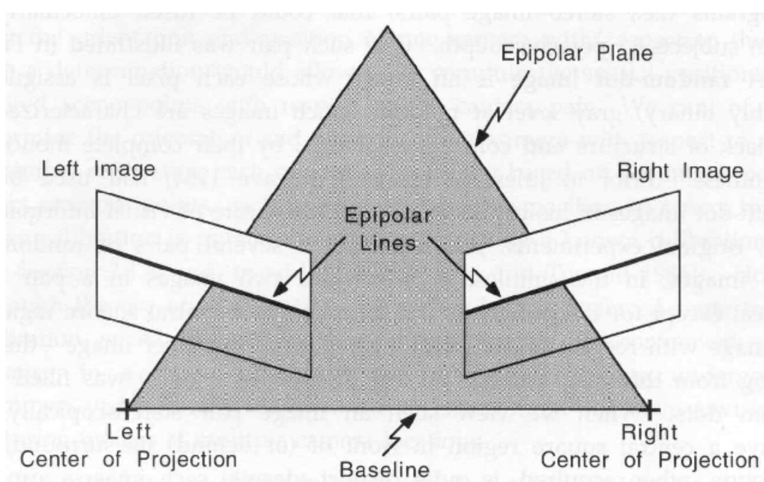
<u>Left epipole:</u> the projection of O_r on the left image plane.

<u>Right epipole:</u> the projection of O_l on the right image plane.

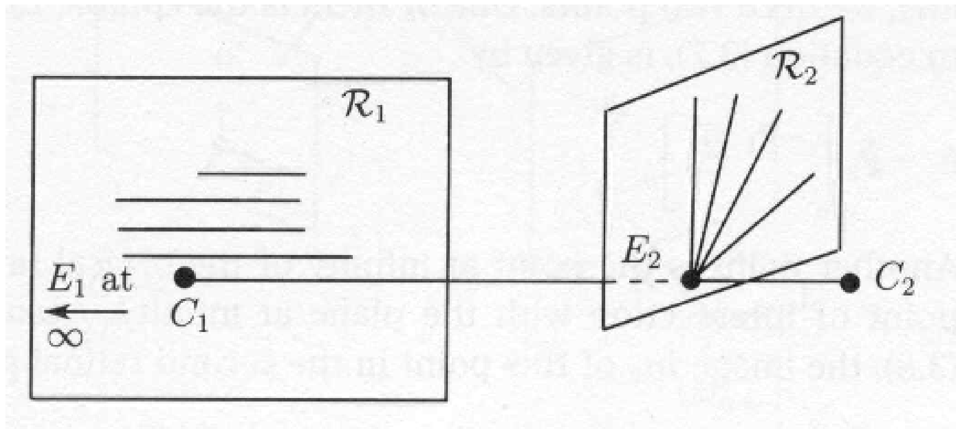
<u>Epipolar plane:</u> the plane defined by P, O_l and O_r.

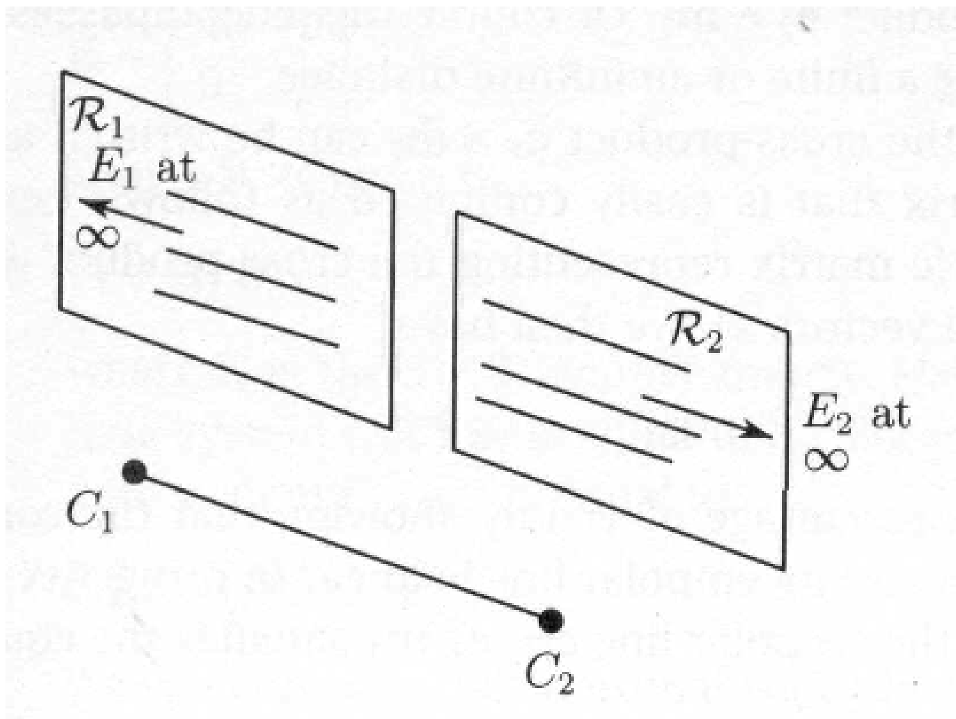
<u>Epipolar line:</u> the intersection of the epipolar plane with the image plane.





Note: if the line through the centers of projection is parallel to one of the image planes, the corresponding epipole is at *infinity*.





• Stereo basics

- The camera frames are related by a translation vector $T = (O_r - O_l)$ and a rotation matrix R.

- The relation between P_l and P_r (projection of P in the left and right frames) is given by

$$P_r = R(P_l - T)$$

- The usual equations of perspective projection define the relation between 3D points and their projections:

$$p_l = \frac{f_l}{Z_l} P_l, \qquad p_r = \frac{f_r}{Z_r} P_r$$

• Epipolar constraint

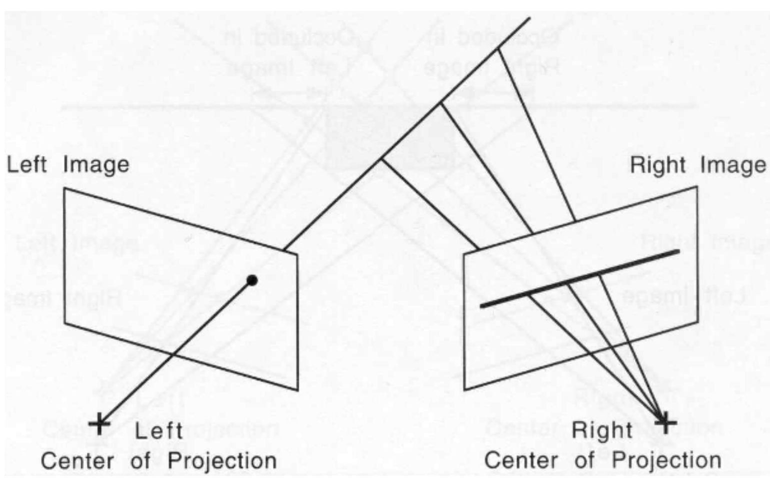
- Given p_l, P can lie anywhere on the ray from O_l through p_l.

- The image of this ray in the right image image is the epipolar line through the corresponding point p_r.

Epipolar constraint: "the correct match must lie on the epipolar line".

- Establishes a *mapping* between points in the left image and lines in the right image and vice versa.

Property: all epipolar lines go through the camera's epipole.



• **Importance of the epipolar constraint**

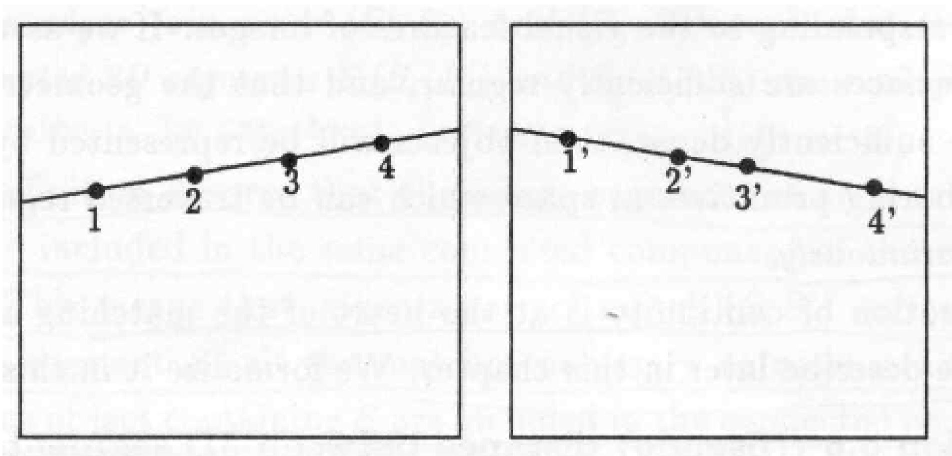
- Corresponding points must lie on conjugate epipolar lines.

- The search for correspondences is reduced to a 1D problem.

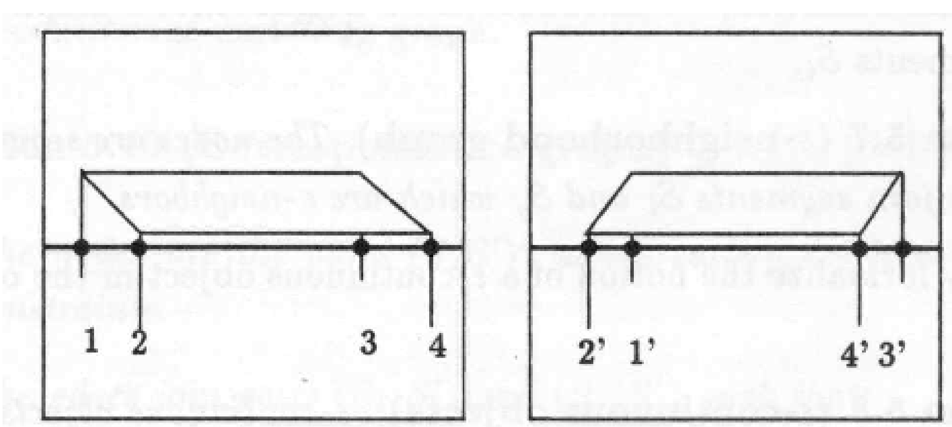
- Very effective in *rejecting false matches* due to occlusion (how?)

• **Ordering**

- Conjugate points along corresponding epipolar lines have the same order in each image.



- *Exception:* corresponding points may not have the same order if they lie on the same epipolar plane and imaged from different sides.



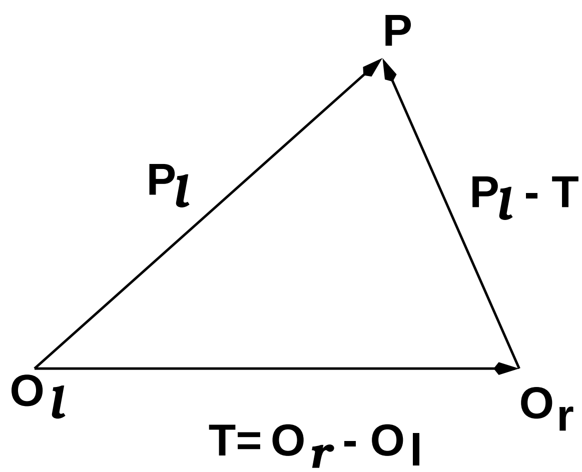
Estimating the epipolar geometry

- How to estimate the mapping between points in one image and epipolar lines in the other?

• The essential matrix, E

- The equation of the epipolar plane is given by the following coplanarity condition (assuming that the world coordinate system is aligned with the left camera):

$$(P_l - T)^T (T \ x \ P_l) = 0$$



- Using $P_r = R(P_l - T)$ we have:

$$(R^T P_r)^T T \ x \ P_l = 0$$

- Expressing cross product as matrix multiplication we have:

$$(R^T P_r)^T S P_l = 0 \quad \text{or} \ P_r{}^T R S P_l = 0 \ \text{where} \quad S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

(the matrix S is always rank deficient, i.e., $rank(S) = 2$)

- The above equation can now be rewritten as follows:

$$\mathbf{P_r^T E P_l = 0} \quad \text{or} \quad \mathbf{p_r^T E p_l = 0}$$

where $E = RS$ is called the *essential matrix*.

- The equation $\mathbf{p_r^T E p_l = 0}$ defines a mapping between points and epipolar lines.

- Properties of the essential matrix:

  (1) encodes info on the extrinsic parameters only
  (2) has rank 2
  (3) its two nonzero singular values are equal

• The fundamental matrix, F

- Suppose that M_l and M_r are the matrices of the intrinsic parameters of the left and right camera, then the pixel coordinates $\bar{p}_l$ and $\bar{p}_r$ of p_l and p_r are:

$$\bar{p}_l = M_l p_l, \quad \bar{p}_r = M_r p_r$$

- Using the above equations and $\mathbf{p_r^T E p_l = 0}$ we have:

$$\mathbf{\bar{p}_r^T F \bar{p}_l = 0}$$

where $F = (M_r^{-1})^T E M_l^{-1} = (M_r^{-1})^T RSM_l^{-1}$ is called the *fundamental matrix*.

- Properties of the fundamental matrix:

  (1) encodes info on both the extrinsic and intrinsic parameters
  (2) has rank 2

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• Computing F (or E): the eight-point algorithm

- We can reconstruct the epipolar geometry by estimating the fundamental matrix from point correspondences only (with no information at all on the extrinsic or intrinsic camera parameters!!).

- Each correspondence leads to a homogeneous equation of the form:

$$\bar{p}_r^T F \bar{p}_l = 0 \quad \text{or}$$

$$x_l x_r f_{11} + x_l y_r f_{21} + x_l f_{31} + y_l x_r f_{12} + y_l y_r f_{22} + y_l f_{32} + x_r f_{13} + y_r f_{23} + f_{33} = 0$$

- We can determine the entries of the matrix F (up to an unknown scale factor) by establishing $n \geq 8$ correspondences:

$$Ax = 0$$

- It turns out that A is rank deficient (i.e., $rank(A) = 8$); the solution is unique up to a scale factor (i.e., proportional to the last column of V where $A = UDV^T$).

Algorithm

Input: n correspondences with $n \geq 8$

1. Construct homogeneous system $Ax = 0$ where A is an nx9 matrix. Suppose $A = UDV^T$ is its SVD.

2. The entries of F are proportional to the components of the last column of V.

Enforcing the constraint $rank(F) = 2$**:** (singularity constraint)

3. compute the SVD of F

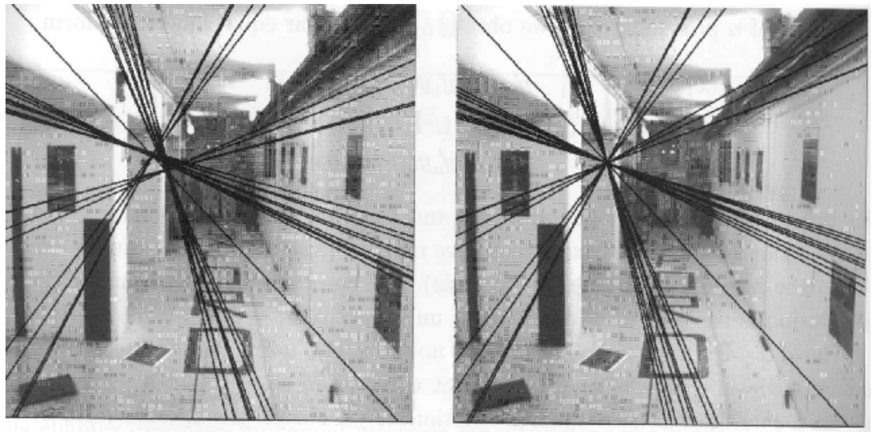
$$F = U_F D_F V_F^T$$

4. Set the smallest singular value equal to 0; Let D'_F be the corrected matrix.

5. The corrected estimate of F, F', is given by

$$F' = U_F D'_F V_F^T$$

Important: we need to normalize the coordinates of the corresponding points, otherwise, A has a very bad condition number which leads to numerical instabilities (see p156).



• Homogeneous (projective) representation of lines (see Appendix A.4)

- A line $ax + by + c = 0$ is represented by the homogeneous vector below (*projective line*):

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- Any vector $k\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ represents the same line.

- Only the ratio of the homogoneous line coordinates is significant:

  * lines can be specified by 2 parameters only (e.g., slope/intercept):

$$y = mx + i$$

  * rewrite $ax + by + c = 0$ in the slope/intercept form:

$$y = -\frac{a}{b}x - \frac{c}{b}$$

  * homogenization rule for lines: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$: $\begin{bmatrix} -a/b \\ -c/b \end{bmatrix}$

- Some properties involving points and lines:

  (1) The point x lies on the line iff $x^T l = 0$
  (2) Two points define a line: $l = p \ x \ q$
  (3) Two lines define a point: $p = l \ x \ m$

<u>Duality:</u> in homogeneous (projective) coordinates, points and lines are dual (we can interchange their roles).

Example1: (0,-1) lies on $2x + y + 1 = 0$

The point (0,-1) is represented by $x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

The line $2x + y + 1 = 0$ is represented by $l = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$x^T l = 0.2 + (-1).1 + 1.1 = 0$$

Example2: find the intersection of $x = 1$ and $y = 1$

The line $x = 1$ is equivalent to $-1x + 1 = 0$ or $l = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The line $y = 1$ is equivalent to $-1y + 1 = 0$ or $l' = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

The intersection point x is

$$x = l \ x \ l' = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example3: find the intersection of $x = 1$ and $x = 10$ (parallel)

The line $x = 1$ is equivalent to $-1x + 1 = 0$ or $l = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The line $x = 10$ is equivalent to $-1x + 10 = 0$ or $l' = \begin{bmatrix} -1 \\ 0 \\ 10 \end{bmatrix}$

The intersection point x is

$$x = l \times l' = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 0 & 10 \end{vmatrix} = \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix}$$

• Finding the epipolar lines

- The equation below defines a mapping between points and epipolar lines:

$$\mathbf{\bar{p}_r^T F \bar{p}_l = 0}$$

<u>Case 1:</u> right epipolar line u_r

the right epipolar line is represented by $u_r = F\,\bar{p}_l$

$\bar{p}_r$ lies on u_r, that is, $\bar{p}_r^T u_r = 0$ or $\mathbf{\bar{p}_r^T F \bar{p}_l = 0}$

<u>Case 2:</u> left epipolar line u_l

$\mathbf{\bar{p}_r^T F \bar{p}_l = 0}$ is equivalent to $\mathbf{\bar{p}_l^T F^T \bar{p}_r = 0}$

the left epipolar line is represented by $u_l = F^T\,\bar{p}_r$

$\bar{p}_l$ lies on u_l, that is, $\bar{p}_l^T u_l = 0$ or $\mathbf{\bar{p}_l^T F^T \bar{p}_r = 0}$

• Locating the epipoles from F (or E)

Case 1: locate $\bar{e}_l$

$\bar{e}_l$ lies on all epipolar lines in the left image, thus, it satisfies the equation:

$$\bar{e}_l^T u_l = 0 \quad \text{or} \quad u_l^T \bar{e}_l = 0 \quad \text{or} \quad \bar{\mathbf{p}}_\mathbf{r}^\mathbf{T} \mathbf{F} \bar{\mathbf{e}}_\mathbf{l} = \mathbf{0}$$

which leads to the following homogeneous system:

$$\mathbf{F} \bar{\mathbf{e}}_\mathbf{l} = \mathbf{0}$$

We can obtain $\bar{e}_l$ by solving the above homogeneous system (the solution $\bar{e}_l$ is proportional to the last column of V of the SVD of F).

Case 2: locate $\bar{e}_r$

$\bar{e}_r$ lies on all epipolar lines in the right image, thus, it satisfies the equation:

$$\bar{e}_r^T u_r = 0 \quad \text{or} \quad \bar{e}_r^T F \bar{p}_l = 0 \quad \text{or} \quad \bar{\mathbf{p}}_\mathbf{l}^\mathbf{T} \mathbf{F}^\mathbf{T} \bar{\mathbf{e}}_\mathbf{r} = \mathbf{0}$$

which leads to the following homogeneous system:

$$\mathbf{F}^\mathbf{T} \bar{\mathbf{e}}_\mathbf{r} = \mathbf{0}$$

The solution is proportional to the last column of V of the SVD of F^T (i.e., same as the last column of U of the SVD of F).

$$F^T = VDU^T$$

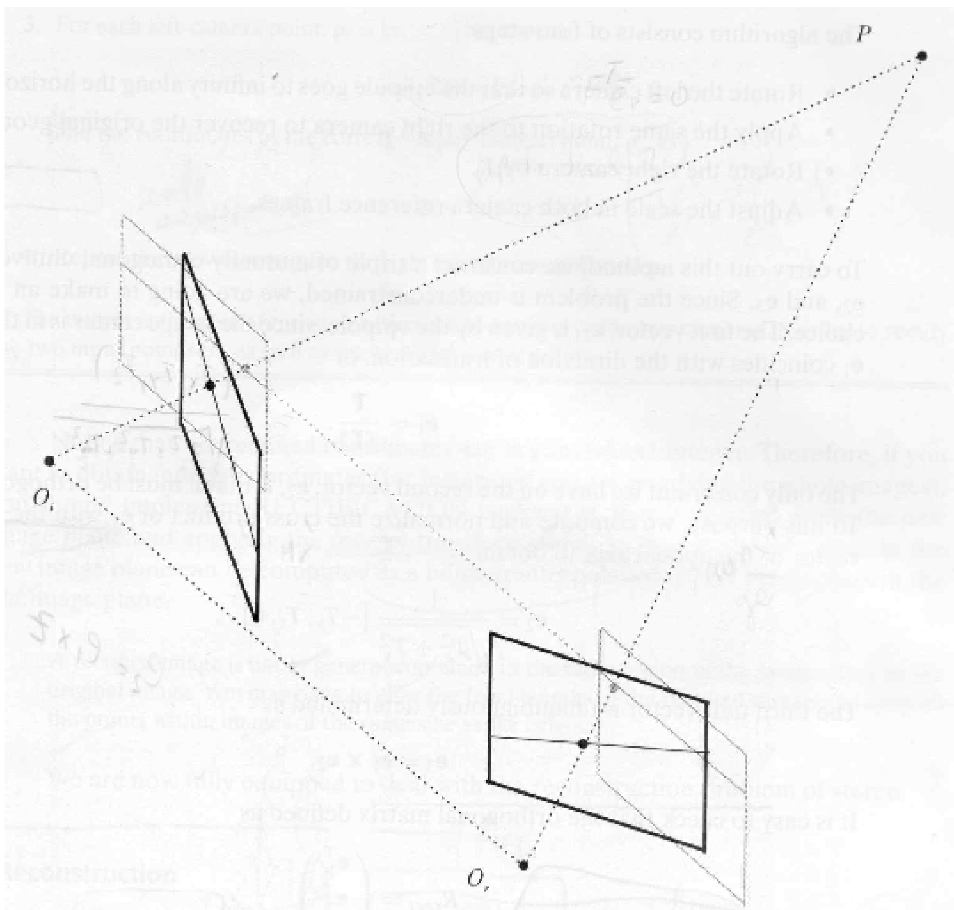
• Rectification

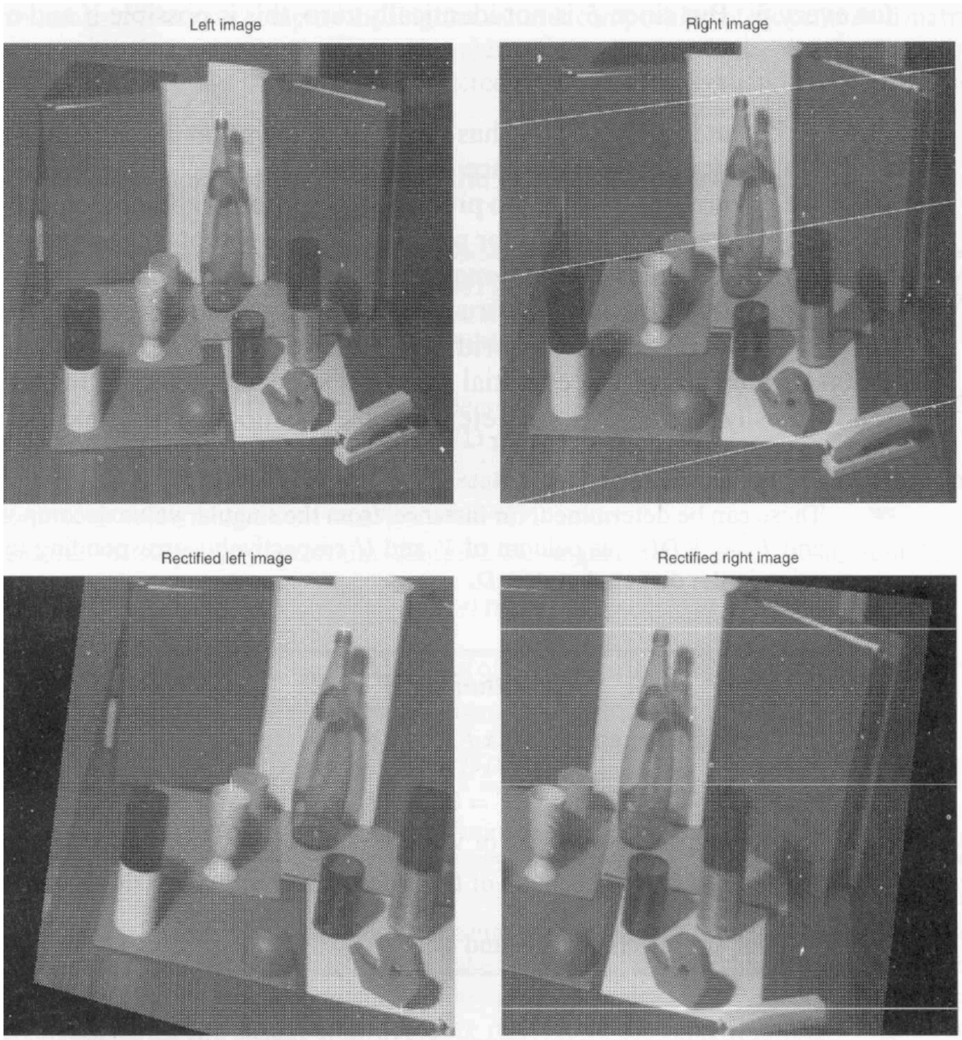
- This is a tranformation of each image such that pairs of conjugate epipolar lines become collinear and parallel to the horizontal axis.

- Searching for corresponding points becomes much simpler for the case of rectified images:

to find the point corresponding to (i_l, j_l) *in the left image,*
we just need to look along the scanline $j = j_l$ *in the right image*

- Disparities between the images are in the x-direction only (no y disparity)





- Main steps (assuming knowledge of the extrinsic/intrinsic parameters):

  (1) Rotate the left camera so that the epipolar lines become parallel to the horizontal axis (epipole is mapped to infinity).

  (2) Apply the same rotation to the right camera to recover the original geometry.

  (3) Rotate the right camera by R

- Let's consider step (1) (the rest are straightforward):

We will construct a coordinate system (e_1, e_2, e_3) centered at O_l.

Aligning the axes of this coordinate system with the axes of the image plane coordinate system yields the desired rotation matrix.

(1.1) e_1 is a unit vector along the vector T

$$e_1 = \frac{T}{\|T\|} = \frac{[T_x, T_y, T_z]^T}{\sqrt{T_x^{\,2} + T_y^{\,2} + T_z^{\,2}}}$$

(1.2) e_2 is chosen as the cross product of e_1 and the z axis

$$e_2 = \frac{e_1 \ x \ [0,0,1]^T}{\|e_1 \ x \ z\|} = \frac{1}{\sqrt{T_x^{\,2} + T_y^{\,2}}}[-T_y, T_x, 0]^T$$

(1.3) choose e_3 as the cross product of e_1 and e_2

$$e_3 = e_1 \ x \ e_2$$

- The rotation matrix that maps the epipoles to infinity is: $R_{rect} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \end{bmatrix}$

- *Note:* rectification can also be done without knowledge of the extrinsic/intrinsic parameters (more complicated).