Review on Eigenvalues and Eigenvectors

• Reading Assignments

H. Anton and C. Rorres, *Elementary Linear Algebra* (Applications Version), 8th edition, John Wiley, 2000 (chapt 7, hard copy).

J. Pricipe et al., *Neural and Adaptive Systems: Fundamentals Through Simulations*, (Appendix A: Elements of Linear Algebra and Pattern Recognition, pp. 590-594, hard copy).

K. Kastleman, *Digital Image Processing*, Prentice Hall, (Appendix 3: Mathematical Background, hard copy).

F. Ham and I. Kostanic. *Principles of Neurocomputing for Science and Engineering*, Prentice Hall, (Appendix A: Mathematical Foundation for Neurocomputing, hard copy)

• Other Books

B. Kolman and D. Hill, *Introductory Linear Algebra with Applications*, 2nd edition, Prentice Hall, 2001.

L. Johnson, R. Riess, and J. Arnold, *Introduction to Linear Algebra*, 4th edition, Addison Wesley, 1998.

Review on Eigenvalues and Eigenvectors

• Definition

- The vector v is an eigenvector of matrix A and the number λ is an eigenvalue of A if:

$$Av = \lambda v \text{ (assuming the non-trivial solution } v = 0\text{)}$$

- The linear transformation implied by A cannot change the direction of the eigenvectors, only their magnitude.

• Characteristic polynomial

- To find the eigenvalues λ of a matrix A, find the roots of the *characteristic polynomial*:

$$det(A - \lambda I) = 0$$

$$\text{Example:} \quad A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$$

$$det(\begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}) = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0 \text{ or } \lambda_1 = 1,\ \lambda_2 = 2$$

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

• Some properties

- Eigenvalues and eigenvectors are only defined for square matrices ($m = n$)

- The eigenvectors are not unique (e.g., if v is an eigenvector, so is kv)

- Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then:

(1) $\sum_i \lambda_i = tr(A)$

(2) $\prod_i \lambda_i = det(A)$

(3) if $\lambda = 0$ is an eigenvalue, then the matrix is not invertible

(4) A and A^2 have the same eigenvectors

(5) if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2

(6) a matrix A with positive eigenvalues is called *positive definite* (the following is true: $x^T Ax > 0$ for every $x \neq 0$)

• Diagonalization

- The problem is finding an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (i.e., P diagonalizes A)

- Consider the matrix $P = [v_1\ v_2 \cdots v_n]$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A (assume they are distinct) and $v_1\ v_2 \cdots v_n$ are the eigenvectors of A:

$$AP = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix} \quad \text{or} \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_1 = 0,\ \lambda_2 = 2,\ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

• Are all *n*x*n* matrices diagonalizable?

- A is diagonalizable if it has n linearly independent eigenvectors (these vectors form a basis too!)

- If A has n distinct eigenvalues, then the corresponding eigevectors are linearly independent.

- In general, the multiplicity of an eigenvalue should be equal to the number of eigenvectors corresponding to this eigenvalue.

• Decomposition

- Let us assume that A is diagonalizable, it's easy to see that:

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix} P^{-1}$$

• The case of symmetric matrices

- The eigenvalues of symmetric matrices are all real, though not necessarily positive.

- The eigenvectors corresponding to distinct eigenvalues are orthogonal.

- Any symmetric *n*x*n* matrix A can be written as:

$$A = VDV^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

- V is an *orthonormal* matrix whose columns are the "normalized" eigenvectors of A (i.e., using *Gram-Schmidt* normalization) and D is a diagonal matrix containing the eigenvalues λ_i of A.

• Vector representation in the eigenvector space of A

- The following linear transformation represents a vector in the space of eigenvectors of A:

$$y_i = V_i^T x$$

• Whitening transformation

$$y_i = VD^{-1/2} x_i$$

- All the eigenvalues after a whitening transformation become identical.

• The case of non-square matrices

- We can extend the results of matrix diagonalization/decomposition to the case of non-square matrices using *Singular Value Decomposition* (SVD).