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Systems of Linear Equations

- **Reading Assignments**

H. Anton and C. Rorres, *Elementary Linear Algebra* (Applications Version), 8th edition, John Wiley, 2000 (1.1, 1.6, 5.5-5.6, hard copy).

K. Kastleman, *Digital Image Processing*, Prentice Hall, (Appendix 3: Mathematical Background, hard copy).

F. Ham and I. Kostanic. *Principles of Neurocomputing for Science and Engineering*, Prentice Hall, (Appendix A: Mathematical Foundation for Neurocomputing, hard copy)

- **Other Books**

B. Kolman and D. Hill, *Introductory Linear Algebra with Applications*, 2nd edition, Prentice Hall, 2001.

L. Johnson, R. Riess, and J. Arnold, *Introduction to Linear Algebra*, 4th edition, Addison Wesley, 1998.

Systems of Linear Equations

• General form

- An arbitrary system of m linear equations in n unknowns can be written as

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{gathered}$$

or in matrix form as $Ax = b$ where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- When $n>m$, the system is called *over-determined*.

- When $n<m$, the system is called *under-determined*.

- A linear system of the from $Ax = 0$ is called *homogeneous*.

• Solving Ax=b (case m=n)

- Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

- Various methods such as *Gauss elimination, Gauss-Jordan elimination, LU-factorization* etc.

• Conditions for solutions of Ax=b (case m=n)

- If A is invertible, $Ax = b$ has exactly one solution:

$$x = A^{-1}b$$

- The following statements are equivalent:

(1) A is invertible
(2) $Ax = 0$ has only the trivial solution
(3) $det(A) \neq 0$
(4) b is in the column space of A

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

(5) $rank(A|b) = rank(A)$ and $rank(A) = n$
(6) The column/row vectors of A are linearly independent.
(7) The column/row vectors of A span R^n

- The system has *no solution* if $rank(A|b) > rank(A)$

- The sustem has *infinitely many solutions* if $rank(A|b) = rank(A)$ and $rank(A) < n$

• What if A is singular?

- See SVD notes ...

• Conditions for solutions of Ax=0 (case m=n)

(1) Has a non-trivial solution iff $rank(A) < n$

(2) Has the trivial solution iff $rank(A) = n$

• Solving Ax=b or Ax=0 (case $m > n$)

See SVD notes ...