Math 181: Review

Physics Parabola

The parabola is used in physics trajectory problems. Estimating the distance and velocity of an accelerating object is done in this problem using the equations of motion. For the vertical position of the rock: $y = v_0 - \frac{1}{2}gt^2$, the velocity is the derivative of this equation, giving $v_y = v_0 - gt$, where v_0 is the initial velocity. Estimating the velocity is the same is estimating the slope of the parabola.

- If a rock is thrown upward on the planet Earth with a velocity of $11m/s$, its height in meters t seconds later is given by $y = 11t4.9t^2$. (Round your answers to two decimal places.) At $t = 1$,

  $$y = 11(1) - 4.91(1)^2 = 6.09.$$

  The average velocity between times 1 and $1 + h$ is

  $$v_{ave} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{[11(1+h) - 4.91(1+h)^2] - 6.09}{h}$$

  $$= \frac{11 + 11h - 4.91 - 9.82h - 4.91h^2}{h} = \frac{1.18h - 4.91h^2}{h} = 1.18 - 4.91h$$

  if $h \neq 0$.
  For example, take $h = 0.5$.
  The average velocity between times 1 and $1 + 0.5$ is

  $$v_{ave} = \frac{y(1.5) - y(1)}{(1.5) - 1} = \frac{[11(1.5) - 4.91(1.5)^2] - 6.09}{0.5}$$

  $$= \frac{5.45}{0.5} = 10.9$$

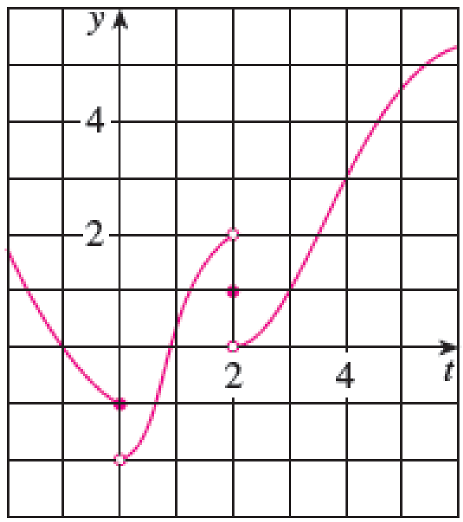
  To find the instantaneous velocity when $t = 1$, we take the limit of the equation as h approaches 0.

  $$\lim_{h \to 0} \frac{y(1+h) - y(1)}{(1+h) - 1} = \lim_{h \to 0}(1.18 - 4.91h) = 1.18$$

Limits

Right and Left-Hand Limits

- For the function g whose graph is given, state the value of each quantity, if it exists.



In this graph, the limit from the left as $t \to 0^-$ does exist, the function is defined at that point, the limit is -1, from the right as $t \to^+$ the limit exists as well, and is -2. Since the limit is different from the right and left, the limit $t \to 0$ does not exist.
At the other discontinuity, even though the function takes on the same *value* at $t = 2$, approaching from the right and the left we have 0 and 2, respectively, so the limit $t \to 2$ does not exist.
The limit is always a tiny bit away (ϵ) from the point, so we look to the right and the left of the limit point to determine if it exists.

Solving Limit Equations

When solving for the limit of a function, we check to make sure the function is in a form where the limit can be found before we 'plug in' the limit number.

- Find the limit $\lim_{h \to 0} \frac{\sqrt{h + a^2} - a}{h}$.
  This is a case where we must multiply by the complex conjugate.

$$\lim_{h\to 0}\frac{\sqrt{h+a^2}-a}{h}\cdot\frac{\sqrt{h+a^2}+a}{\sqrt{h+a^2}+a}=\lim_{h\to 0}\frac{h+a^2-a^2}{h\cdot(\sqrt{h+a^2}+a)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+a^2}+a}=\lim_{h\to 0}\frac{1}{2a}$$

- Find the limit $\lim_{x\to b}\frac{x^2-(b+1)x+b}{x-b}$.
  In this case cancel the denominator.

$$\lim_{x\to b}\frac{x^2-bx-x+b}{x-b}=\lim_{x\to b}\frac{x\cdot(x-b)+(-1)(x-b)}{x-b}=\lim_{x\to b}x-1=b-1$$
$$\lim_{h\to b}$$

- Find the limit $\lim_{x\to -c}\frac{\frac{1}{c}+\frac{1}{x}}{x+c}$.
  Here we will multiply by $\frac{cx}{cx}$ to remove the fractions in the numerator (multiplication by 1).

$$\lim_{x\to -c}\frac{\frac{1}{c}+\frac{1}{x}}{x+c}\cdot\frac{cx}{cx}=\lim_{x\to -c}\frac{x+c}{cx\cdot(x+c)}=\lim_{x\to -c}\frac{1}{cx}=\frac{1}{-c^2}$$

Continuity

- Find the values of a and b to make f continuous everywhere.

$$f(x)=\begin{cases}\frac{x^2-4}{x-2} & \text{if } x<2\\ ax^2-bx+3 & \text{if } 2\le x<3\\ 2x-a+b & \text{if } x\ge 3\end{cases}$$

At $x=2$:

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}\frac{x^2-4}{x-2}=\lim_{x\to 2^-}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2^-}(x+2)=2+2=4$$
$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(ax^2-bx+3)=4a-2b+3$$

We must have
$$4a-2b+3=4, or 4a-2b=1\ \mathbf{(1)}.$$

At $x=3$:
$$\lim_{x\to 3^-}f(x)=\lim_{x\to 3^-}(ax2-bx+3)=9a-3b+3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x - a + b) = 6 - a + b$$

We must have $9a - 3b + 3 = 6 - a + b$, or $10a - 4b = 3$ **(2)**. Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{aligned} -8a + 4b &= -2 \\ 10a - 4b &= 3 \\ 2a &= 1 \end{aligned}$$

So $a = 1/2$. Substituting $1/2$ for a in **(1)** gives us -2$b = -1$, so $b = 1/2$ as well. Thus, for f to be continuous on $(-\infty, \infty), a = 1/2$ and $b = 1/2$.

Limits tending towards Infinity

- Find the limit, if it exists.

$$\lim_{x\to\infty} \frac{\sqrt{4x^6 - x}}{x^3 + 7}$$

In this case, we divide by the largest power of x in the numerator and in the denominator. The largest power is x^3 since $\sqrt{x^6} = x^3$ for $x > 0$.

$$\lim_{x\to\infty} \frac{\sqrt{4x^6 - x}/x^3}{(x^3 + 7)/x^3} = \lim_{x\to\infty} \frac{\sqrt{(4x^6 - x)/x^6}}{1 + 7/x^3}$$

$$= \frac{\lim_{x\to\infty} \sqrt{4 - 1/x^5}}{\lim_{x\to\infty} 1 + 7/x^3} = \frac{\sqrt{\lim_{x\to\infty} 4 - \lim_{x\to\infty}(1/x^5)}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(7/x^3)} = \frac{\sqrt{4+0}}{1+0} = 4$$

- Find the limit, if it exists.

$$\lim_{x\to\infty}(\sqrt{64x^2 + x} - 8x)$$

In this case, we both multiply by the complex conjugate and divide by the highest power of x, which is x since $\sqrt{x^2} = x$ for $x > 0$.

$$\lim_{x\to\infty} \frac{(\sqrt{64x^2 + x} - 8x)(\sqrt{64x^2 + x} + 8x)}{\sqrt{64x^2 + x} + 8x} = \lim_{x\to\infty} \frac{(64x^2 + x) - 64x^2}{\sqrt{64x^2 + x} + 8x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{64x^2 + x} + 8x} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{x/x}{\sqrt{64x^2/x + x/x^2} + 8x/x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{64 + 1/x + 8}} = \lim_{x\to\infty} \frac{1}{\sqrt{64} + 8} = \frac{1}{8+8} = \frac{1}{16}$$

Derivatives

The derivative of a function represents the rate of increase or decrease of the function. If the value of x changes by a certain amount, the derivative will tell you how much the y value also has to change.
If a function is differentiable at a point, then it has to be continuous at that point. (it's because derivative can only be taken on smooth lines, imagine taking the derivative of a discontinuity, it wouldn't work!)

- Find an equation of the tangent line to the curve at the given point.

$$y = x^3 - 3x + 1 \ (4, 53)$$

Here we can just use the definition:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Where $f(x) = x^3 - 3x + 1$ and $P(4, 53)$.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(4+h)^3 - 3(4+h) + 1 - 53}{h}$$

$$= \lim_{h \to 0} \frac{64 + 48h + 12h^2 - 12 - 3h - 52}{h} = \lim_{h \to 0} \frac{45h + 12h^2 + h^3}{h} = \lim_{h \to 0}(45 + 12h + h^2) = 45$$