Math 181: Review

Derivatives

Tangents and Normals

Finding the tangent lines and normals requires you to have two pieces of information: a point on the function $(x, f(x))$ and the slope of the line at that point. If you have the slope of the tangent, then the slope of the normal is the negative reciprocal, since they are perpendicular, and vice versa. Find the slope at a point by putting x into the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or by finding the derivative using formulas if f is an elementary function,

$$\frac{d}{dx} a^{b \cdot x} = (a \cdot b) x^{(b-1)}$$

Since tangents and normals are lines, the slope is the same everywhere. The function **must** be *continuous* at a point to take the derivative there. Now use the point-slope form for a line, lets say we have point (a, b), the equation for the tangent line is

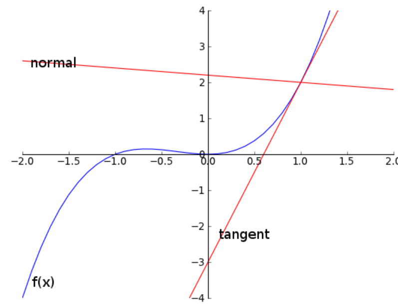
$$y = f'(a) \cdot (x - a) + b$$

just change the slope to get the normal line at the same point.
Notice that this is the instantaneous slope at point (a, b) (the tangent line), where if we were approximating over an interval $[A, B]$ (the secant line), then our definition gives the slope over an interval,

$$f'([A, B]) = \frac{f(B) - f(A)}{B - A}$$

where $h = B - A$ (we are taking the first value in the interval, A as our x value and $B - A$ as our h value to grab the whole interval, i.e. $(f(A + (B - A) - f(A))/B - A)$.

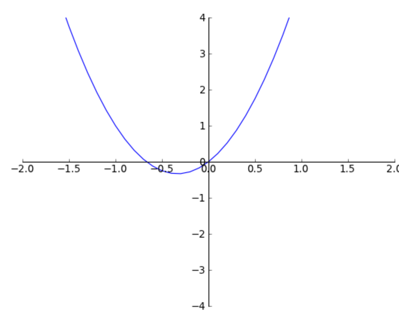
Physics Note: The velocity is the derivative of position, the acceleration is the derivative of velocity.



Normal $= 5(x-1)+2$, and tangent $= (-1/5)(x-1)+2$
to $f(x) = x^3 + x^2$ at $(1,2)$

Graph the Derivative

Note where the function is positive and negative, and where it is 0. Where the slope of the function is 0 (horizontal) the graph of the derivative will hit the 0 (x-axis). If the of the function is increasing, the values of the derivative will be positive (above the x-axis), if it is decreasing, the values of the derivative will be negative. If the rate of change (slope) of the function is increasing (with y-values getting large very fast) or decreasing (y-values not changing so fast), then the graph of the derivative will do exactly this.
In the above graph, the slope starts out positive, then crosses 0 and is negative, crosses again and is positive, going from steep to shallow to steep again.



$f(x) = 3x^2 + 2x$

Domains

The domain of a function and the domain of the derivative might not be the same.

$$g(x) = \sqrt{9-x}$$

has domain $(-\infty, 9]$, since we are ok with a function being zero.
The derivative,

$$g'(x) = \frac{-1}{2\sqrt{9-x}}$$

cannot include the number 9, as it would put 0 in the denominator, so has domain $(-\infty, 9)$.
To find the derivative of g we need to use the definiton. In order to cancel the h in the denominator we need to rationalize the square root by multiplying by its conjugate: (Multiply by the conjugate divided by itself to equal 1)

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h} \cdot \frac{\sqrt{9-(x+h)} + \sqrt{9-x}}{\sqrt{9-(x+h)} + \sqrt{9-x}}$$

$$= \lim_{h\to 0} \frac{\sqrt{9-(x+h)}^2 + \sqrt{9-(x+h)} \cdot \sqrt{9-x} - \sqrt{9-(x+h)} \cdot \sqrt{9-x} - \sqrt{9-x}^2}{h \cdot [\sqrt{9-(x+h)} + \sqrt{9-x}]}$$

Notice when we FOIL the inside and outside cancel,
the first and last are squared and $\sqrt{x}^2 = x$

$$= \lim_{h\to 0} \frac{[9-(x+h)] - (9-x)}{h \cdot \sqrt{9-(x+h)} + \sqrt{9-x}}$$

$$= \lim_{h\to 0} \frac{-h}{h \cdot \sqrt{9-(x+h)} + \sqrt{9-x}}$$

Now we can cancel the h in the denominator.

$$= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)} + \sqrt{9-x}}$$

Take the limit!

$$= \frac{-1}{2\sqrt{9-x)}}$$

Distribution Laws: Addition and Subtraction

The derivative can be distributed through addition and subtraction, just like limits. **BUT** derivatives are not *closed* under multiplication and division. You cannot distribute the limit into functions that are multiplied or divided.

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

i.e.

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) \neq \frac{d/dxe^x}{d/dxx}$$

Note: Don't forget the derivative of any *constant* (just any number) is 0.
In general, it can help to simplify the function before taking the derivative, i.e. it is easier to take the derivative of

$$e^{-x}$$

than

$$\frac{1}{e^x}$$

- Find the derivative of

$$y = \frac{2x^2 + 8x + 4}{\sqrt{x}}$$

$$\begin{aligned} y' &= \frac{d}{dx}(2x^2 + 8x + 4) \cdot x^{-1/2} \\ &= \frac{d}{dx}(2x^{2-1/2} + 8x^{1-1/2} + 4x^{-1/2}) \\ &= \frac{d}{dx}(2x^{3/2} + 8x^{1/2} + 4x^{-1/2}) \\ &= 2\left(\frac{3}{2}\right)x^{3/2-1} + 8\left(\frac{1}{2}\right)x^{1/2-1} + 4\left(-\frac{1}{2}\right)x^{-1/2-1} \\ &= 3x^{1/2} + 4x^{-1/2} - 2x^{-3/2} \end{aligned}$$

We can simplify since

$$x^{-3/2} = x^{-1/2-2/2}$$

Our common denominator is $x^{-3/2}$. Remember $\frac{1}{x^{-a}} = x^a$. So,

$$\begin{aligned} &3x^{1/2} \cdot \frac{x^{-3/2}}{x^{-3/2}} + 4x^{-1/2} \cdot \frac{x^{-1}}{x^{-1}} - 2x^{-3/2} \\ &= 3x^{1/2+3/2} \cdot x^{-3/2} + 4x^1 \cdot x^{-1/2-1} - 2x^{-3/2} \\ &= x^{-3/2}(3x^2 + 4x - 2) \end{aligned}$$

The Exponential Function

The exponential function is described by the number e, defined by the limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

The number e has the property that if you have a *natural* log function, you can get rid of the log by making it a power of e. i.e.
You have $\ln y = x$. Make it a power of e to find the function.

$$e^{\ln y} = y = e^x$$

Now we have our function $y = e^x$. The natural log is a special case of the normal log function, we use it along with the exponential function since they cancel out. We are using the property of all logs to pull exponents down from an expression. Say you have a^x, take the log to pull down the x.

$$\log a^x = x \log a$$

Distribution Laws: Multiplication and Division

Product Rule

$$(fg)'(x) = fg' + gf'$$
$$first - dee - last\ plus\ last - dee - first$$

More functions! Just take the derivative of one, and add them together. Make sure you have a derivative for each function.

$$(fgh)'(x) = f'gh(x) + fg'h(x) + fgh'(x)$$

- Find the first and second derivatives of $x^{5/2}e^x$.
  Here, we take $f = x^{5/2}$ and $g = e^x$. Plug in the formula.

  $$f'(x) = \frac{5}{2}x^{3/2}e^x + x^{5/2}e^x$$

  We take the derivative again of each term (distribute through the addition)

  $$f''(x) = \left(\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)x^{1/2}e^x\right) + \left(\frac{5}{2}x^{3/2}e^x\right) + \left(\left(\frac{5}{2}x^{3/2}e^x + x^{5/2}e^x\right)\right)$$
  $$= \left[\left(\frac{15}{4}\right)x^{1/2}e^x\right] + \left[\frac{10}{2}x^{3/2}e^x + x^{5/2}e^x\right]$$

- Differentiate.
$$f(x) = 6\sqrt{x}\sin x$$
We don't worry about the 6, it's just a constant.
$$\begin{aligned} f'(x) &= 6\sqrt{x} \cdot d/dx \sin x + d/dx\sqrt{x} \cdot \sin x \\ &= 6\sqrt{x} \cdot \cos x + x^{-3/2}\sin x \end{aligned}$$

- Differentiate.
$$y = 3x^2 \sin x \tan x$$
There are three functions here, so we will need to add 3 terms, with a derivative in each.
$$\begin{aligned} y' &= d/dx(3x^2) \cdot \sin x \tan x + 3x^2 \cdot d/dx(\sin x) \cdot \tan x + 3x^2 \cdot \sin x \cdot d/dx(\tan x) \\ &= 6x \cdot \sin x \tan x + 3x^2 \cdot \cos x \cdot \tan x + 3x^2 \cdot \sin x \cdot \sec^2 x \end{aligned}$$
Notice that since $\tan x = \frac{\sin x}{\cos x}$, we can simplify by pulling out a $3x \sin x$
$$= 3x \cdot (2 \cdot \sin x \tan x + x \cdot \cos x \cdot \frac{\sin x}{\cos x} + x \cdot \sin x \cdot \sec^2 x$$
The cosines cancel.
$$= 3x \sin x \cdot (2 \cdot \tan x + x + x \cdot \sec^2 x$$

Quotient Law

$$\left(\frac{f}{g}\right)'(x) = \frac{gf' - fg'}{g^2}$$

$$low - dee - high\ minus\ high - dee - low,\ draw\ the\ line\ and\ square\ below$$

- Differentiate.
$$y = \frac{x^3}{2 - x^2}$$
First we write out the entire equation using the formula.
$$y' = \frac{(2 - x^2) \cdot d/dx(x^3) - (x^3) \cdot d/dx(2 - x^2)}{(2 - x^2)^2}$$
Take the derivatives.
$$\frac{(2 - x^2) \cdot 3x^2 - (x^3) \cdot (-2x)}{(2 - x^2)^2}$$
(Don't forget the derivative distributes through $(2 - x^2)$ so $d/dx(2 - x^2) = -2x$. Simplify.
$$\frac{6x^2 - 3x^4 - (-2x^4)}{(2 - x^2)^2} = \frac{x^2(6 - x^2)}{(2 - x^2)^2}$$