Math 181: Review

More Derivatives

Trigonometric Functions

If you cannot remember the derivative of a trigonometric function. You may need to derive it. Just remembering a few simple trigonometric laws can make a big difference.

- Find
$$\frac{d}{dx}\cot x$$
Use the fact that $\cot x = \frac{1}{\tan x}$ and use the quotient rule.
$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{1}{\tan x} = \frac{\tan x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\tan x}{\tan^2 x}$$

- Find
$$\frac{d}{dx}\sec x$$
Use the fact that $\sec x = \frac{1}{\cos x}$ and use the quotient rule.
$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\cos x}{\cos^2 x}$$
$$= \frac{-(-\sin^2 x)}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$$

Chain Law

$$(f(g))'(x) = f'(g)(x) \cdot g'(x)$$
$$(f(g(h))'(x) = f'(g(h))(x) \cdot g'(h)(x) \cdot h'(x)$$

You can work from the inside-out, or from the outside-in, multiplying each derivative you come across. Leave everything else alone except for the specific function you are taking the derivative of (you can replace everything else by u.

- Find the derivative.
$$f(x) = (x^4 + 3x^2 - 4)^8$$
Only two functions here.
$$f'(x) = \frac{d}{dx}(x^4 + 3x^2 - 4)^6 \cdot \frac{d}{dx}(x^4 + 3x^2 - 4)$$

Lets replace the inner function by u to make it more clear which one is inside and which one is outside.

$$= \frac{d}{du}(u^6) \cdot \frac{d}{dx}(x^4 + 3x^2 - 4)$$

$$= 6u^5 \cdot (4x^3 + 6x)$$

Don't for get to put u back.

$$= 6 \cdot (x^4 + 3x^2 - 4)^5 \cdot (4x^3 + 6x)$$

- Find the derivative.

$$y = \cos(a^8 + x^8)$$

Again, only two functions.

$$f'(x) = \frac{d}{dx}\cos(a^8 + x^8) \cdot \frac{d}{dx}(a^8 + x^8)$$

Replace by u, why not?

$$= \frac{d}{du}\cos u \cdot \frac{d}{dx}(a^8 + x^8)$$

Here, a^8 is a constant since our variable is x, so will go to 0.

$$= -\sin u \cdot 8x^7)$$

$$= -\sin(a^8 + x^8) \cdot 8x^7)$$

- Differentiate.

$$f(x) = x \sin 2x$$

Here we have 3 functions, $x, \sin x$, and $2x$, we will need chain rule and product rule.

$$f'(x) = \frac{d}{dx}(x \sin 2x) = \frac{d}{dx}(x) \cdot \sin 2x + x \cdot \frac{d}{dx}\sin 2x$$

Product rule.

$$= 1 \cdot \sin 2x + x \cos 2x \cdot \frac{d}{dx}(2x)$$

Chain rule.

$$= \sin 2x + x \cos 2x \cdot 2$$

- Find the derivative.

$$y = 3xe^{-kx}$$

Don't forget when the exponent of an exponential is a function! Again we have 3 functions. Use product and chain rule.

$$y' = \frac{d}{dx}(3x) \cdot e^{-kx} + 3x \cdot \frac{d}{dx}(e^{-kx})$$

Product rule.

$$= 3 \cdot e^{-kx} + 3x \cdot e^{-kx} \cdot \frac{d}{dx} - kx$$

Chain rule.

$$= 3 \cdot e^{-kx} + 3x \cdot e^{-kx} \cdot -k$$

Simplify!

$$= 3e^{-kx}(1 - kx)$$

- **Special Cases** If you need to take the derivative of a function raised to the power of x, i.e. a^x, you will need to use the exponential and natural log along with the chain law.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = a^x \ln a$$

Don't forget that when you take the derivative of a compound exponent, the exponent counts as a function too.

$$\frac{d}{dx}(a^{bx}) = \frac{d}{dx}(e^{bx \ln a}) \cdot \frac{d}{dx}(bx \ln a) = a \cdot a^b x \cdot b \ln a$$

Implicit Differentiation

So far we have just taken the derivative for one variable, which represents the rate of change. But what if our function represents something that changes in more than one dimension?
Lots of physical systems are dependent on two variables. A rubber ball deforms if it is thrown, the deformation of the ball is dependent on the velocity. An explosion is described by the amount of heat it produces, and how quickly the heat travels in space.
In the simplest terms, we need two variables to describe what is happening, which creates a *contour* (curvy) map in 3D.
In order to take the derivative of an implicit function, we skip the hard part (solving for y), which would add a lot of unecessary computation. IInstead, we can assume

that y is a dependent function of x, and make up a rule to take the derivative of both sides of the equation.

Implicit Chain Rule

$$\frac{d}{dx}f(y) = f'(y) \cdot y'$$

What we are doing is assuming that $y = y(x)$, so that

$$\frac{dy(x)}{dx} = \frac{d}{dy}y(x) \cdot \frac{dy}{dx} = f'(y)(x) \cdot y'(x)$$

In the end, we can solve for our y' to find the solution. Notice that the dys cancel to give us $\frac{d}{dx}y(x)$.

- Find $\frac{dy}{dx}$.

  $$x^8 + y^5 = 3$$

  We can take our x derivative normally.

  $$8x^7 + \frac{d}{dx}(y^5) = 0$$

  Use implicit chain rule.

  $$8x^7 + 5y^4 \cdot \frac{dy}{dx}$$

  Solve for y'.

  $$\frac{dy}{dx} = \frac{8x^7}{5y^4}$$

- Use implicit differentiation to find an equation of the tangent line to the given point.

  $$y \sin 12x = x \cos 2y, \;\; (\pi/2, \pi/4)$$

  We need find the slope of the line at the given point to complete the equaiton of a line. We are going to need product rule, chain rule *and* implicit chain rule.

  $$\frac{d}{dx}(y) \sin 12x + y\frac{d}{dx}\sin 12x = \frac{d}{dx}(x)\cos 2y + x\frac{d}{dx}\cos 2y$$

  Using product rule.

  $$\frac{d}{dy}y \cdot \frac{dy}{dx} \cdot \sin 12x + y \cdot \frac{d}{dx}\sin 12x = 1 \cdot \cos 2y + x\frac{d}{dx}\cos 2y$$

  Using implicit chain rule.

  $$1 \cdot \frac{dy}{dx} \cdot \sin 12x + y \cdot \cos 12x \cdot \frac{d}{dx}12x = 1 \cdot \cos 2y + x \cdot (-\sin 2y) \cdot \frac{d}{dx}2y$$

Using chain rule.

$$\frac{dy}{dx} \cdot \sin 12x + y \cdot \cos 12x \cdot 12 = \cos 2y + x \cdot (-\sin 2y) \cdot \frac{d}{dy} 2y \cdot \frac{dy}{dx}$$

Using implicit chain rule again.

$$\frac{dy}{dx} \cdot \sin 12x + y \cdot \cos 12x \cdot 12 = \cos 2y + x \cdot (-\sin 2y) \cdot 2 \cdot \frac{dy}{dx}$$

Now we are done differentiating, and can solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} \cdot \sin 12x - x \cdot (-\sin 2y) \cdot 2 \cdot \frac{dy}{dx} = \cos 2y - y \cdot \cos 12x \cdot 12$$

$$\frac{dy}{dx} = \frac{\cos 2y - 12\cos 12x}{\sin 12x + x \sin 2y}$$

We plug in our point to get the slope. Before we only needed the x value which gave us $y = f'(x)$ where y was the slope!
Remember that $\cos 1 = \cos(\pi/2n) = 0$ and $\sin 0 = \sin(2n\pi) = 0$.

$$= \frac{-3\pi \cdot 1 + 0}{0 + \pi \cdot 1} = -3$$

Using the point-slope form of a line, we have

$$y = -3(x - \pi/2) + \pi/4$$