Math 181: Review

More Derivatives

Log Rules

1. $$\log_b(ax) = \log_b(a) + \log_b(x)$$

2. $$\log_b(a/x) = \log_b(a) - \log_b(x)$$

3. $$\log_b(x/a) = \log_b(x) - \log_b(a)$$

4. $$\log_b(x^a) = a \cdot \log_b(x)$$

The log function base e is the natural log.

$$\log_e(x) = \ln(x)$$

Differentiating with Log Functions

- Differentiate.

  $$f(x) = 9x\ln(4x) - 9x$$

  Here we will need product and chain rules.

  $$f'(x) = \frac{d}{dx}9x \cdot \ln(4x) + 9x \cdot \frac{d}{dx}\ln(4x) - \frac{d}{dx}9x$$

  Product rule.

  $$= 9 \cdot \ln(4x) + 9x \cdot \frac{1}{4x}\frac{d}{dx}4x - 9$$

  We need chain rule for the $4x$ inside the log.

  $$= 9 \cdot \ln(4x) + 9x \cdot 4\frac{1}{4x} - 9$$

  Simplify.

  $$= 9 \cdot \ln(4x) + 9 - 9$$
  $$= 9 \cdot \ln(4x)$$

- Differentiate.

$$g(x) = \ln(x\sqrt{x^2 - 11})$$

Here we can use log rule 1 to make the function easier to differentiate. (You can use product rule as well, but using log rules when you see them will save time).

$$g(x) = \ln(x) + \ln((x^2 - 11)^{1/2})$$

Now we differentiate.

$$\begin{aligned} g'(x) &= \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2 - 11} \cdot \frac{d}{dx}(x^2 - 11) \\ &= \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 - 11} \end{aligned}$$

Simplify.

$$\begin{aligned} &= \frac{1}{x} + \frac{x}{x^2 - 11} \\ &= \frac{x^2 - 11}{x(x^2 - 11)} + \frac{x^2}{x(x^2 - 11)} \\ &= \frac{2x^2 - 11}{x(x^2 - 11)} \end{aligned}$$

Word Problems

Interpreting word problems you will need to find out what the problem is asking you to find, and what it already gives you. You will need:

1. The rate of change with respect to the independent variable (k).
2. An expression for the value of the function with respect to the derivative.

$$y = Ce^{kt}$$

3. The value that the function takes at a certain point.
4. Or.. the point at which the function has a certain value.

Pay attention to units! What measurement are they using for time, currency, volume?

Exponential Growth/Decay

- A sample of a radioactive substance decayed to 96.5% of its original amount after a year.
  What is the half-life of the substance?
  Say that the substance starts with mass Ckg. So, $y(0) = C$ at the start of the year. Using our formula, we know $y(t) = Ce^{kt}$. We also know the constant associated with the rate of change, since $y(1) = 0.965C$, we can replace the t with 1 to get $Ce^k = 0.965C$ so $\ln e^k = \ln 0.965$. They want to know the half-life, which is when the original amount of the substance C is halved.
  We need to solve for t when $Ce^{ek} = \frac{1}{2}C$.

$$Ae^(\ln 0.965)t = \frac{1}{2}C$$

$$\ln e^(\ln 0.965)t = \ln\frac{1}{2}$$

$$(\ln 0.965t) = \ln\frac{1}{2}$$

$$t = -\frac{\ln 2}{\ln 0.965}$$

Volume Equations

Cylinder

$$b \cdot h = \pi r^2 h$$

Pyramid

$$\frac{1}{3}b \cdot h$$

Cone

$$\frac{1}{3}b \cdot h = \frac{1}{3}\pi r^2 h$$

Sphere

$$\frac{4}{3}\pi r^2$$

- The radius of a sphere is increasing at a rate of $5mm/s$. How fast is the volume increasing when the diameter is $80mm$?
  In this case we are looking for dV/dt, but our rate of change is in terms of r. Use implicit chain rule on r.

$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}r^3$$

$$= \frac{4}{3}\pi \cdot \frac{d}{dr}r^3 \cdot \frac{dr}{dt}$$
$$= \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

We know from the problem that $dr/dt = 5$. Don't forget to divide the diameter by 2 to get the radius.

$$= \frac{4}{3}\pi \cdot 3(40)^2 \cdot 5$$
$$= 32000\pi mm^3/s$$

Don't forget volume is measured in cubic millimeters, and that since we found the rate of change with respect to time, it is per second.

Linearization

In Review Sheet 2, we used the definition of the derivative to find slope of the tangent line at a certain point. We can now call it linearization. We have done this for the tangent line (linearization), the normal line (where the slope is the negative reciprocal), and for intervals (where h is a finite value).
Since tangents are lines, the slope is the same everywhere. Remember that the point-slope form for a line for point (a, b) is $y = m(x - a) + b$. The equation for the tangent line is

$$y = f'(a) \cdot (x - a) + f(a)$$

for a point $(a, f(a))$.

Differentials

The differential is the variable dx or dy that we have been using in our computations to represent the rate of change of each variable. i.e.

$$dy = f(x + h) - f(x)$$

We can see that dy depends on how much x has changed. If the function does not change with respect to its independent variable (normally time), then $dy = 0$, implying that the function is constant (a flat line).
It is important to realize that they are variables representing real values even though

those values are just snapshots of an evolving system.
Remember for example, implicit chain rule:

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \cdot y'$$

Or we can write it specifying that we are using an implicit variable x:

$$\frac{d}{dx}f(y)(x) = \frac{d}{dy}y(x) \cdot \frac{dy}{dx}$$

The trick is that we multiplied by 1, we multiplied by $\frac{dy}{dy}$.

A differential is a variable, and we can solve for it, but remember it is defined by the rate of change of the related independent variable.