Review on Matrices

- **Matrix multiplication**

$$\begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & b_{1p} \\ b_{21} & b_{22} & . & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{q1} & b_{q2} & . & b_{qp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & c_{1p} \\ c_{21} & c_{22} & . & c_{2p} \\ \cdots & \cdots & c_{ij} & \cdots \\ c_{m1} & c_{m2} & . & c_{mp} \end{bmatrix}$$

(A is an m x n matrix, B is an q x p matrix, C will be a m x p matrix (n=q))

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

- **Properties of matrix multiplication**

$$A(B+C) = AB + AC$$

$$AI = IA = A, \text{ where } I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & . & 1 \end{bmatrix}$$

$$AB \neq BA$$

- **Determinants**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

<u>m x m matrices:</u> $det(A) = \sum_{j=1}^{m} (-1)^{j+k} a_{jk} det(A_{jk})$, for any k: $1 \leq k \leq m$

Property: $det(AB) = det(A)det(B)$

- **Matrix transpose**

$$A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix}, \ A^T = \begin{bmatrix} a_{11} & a_{21} & . & a_{m1} \\ a_{12} & a_{22} & . & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & . & a_{mn} \end{bmatrix}$$

Property: $(AB)^T = B^T A^T$

- **Vector transpose**

If $v = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_m \end{bmatrix}$, then $v^T = [v_1 \ v_2 \cdots v_m]$

- **Defining dot product using vector transpose**

$$v = (v_x, v_y), \ w = (w_x, w_y)$$

$$v.\,w = v_x w_x + v_y w_y \text{ or}$$

$$v.\,w = [v_x \ v_y] \begin{bmatrix} w_x \\ w_y \end{bmatrix} = v^T w$$

- **Matrix inverse**

- The inverse A^{-1} of matrix A has the property: $AA^{-1} = A^{-1}A = I$

- A^{-1} exists only if $det(A) \neq 0$

- **Orthogonal/orthonormal matrices**

- Consider the vectors formed by the rows (or columns) of matrix A:

$$u_1 = [a_{11}\ a_{12} \cdots a_{1n}]$$
$$u_2 = [a_{21}\ a_{22} \cdots a_{2n}]$$
$$\ldots$$
$$u_m = [a_{m1}\ a_{m2} \cdots a_{mn}]$$

Property (1): $u_k . u_k = 1$ or $\|u_k\| = 1$, for every k

Property (2): $u_j . u_k = 0$, for every $j \neq k$ (u_j is perpendicular to u_k)

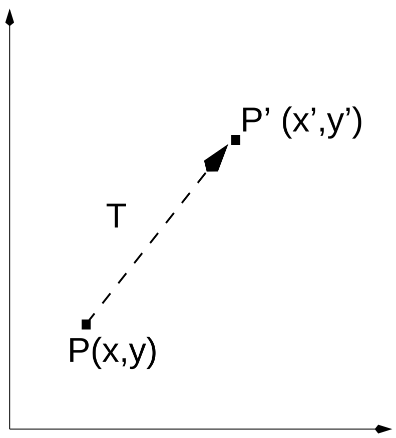
A is orthonormal if both (1) and (2) are satisfied

A is orthogonal if only (2) is satisfied

2D Geometrical Transformations

- **Translation**

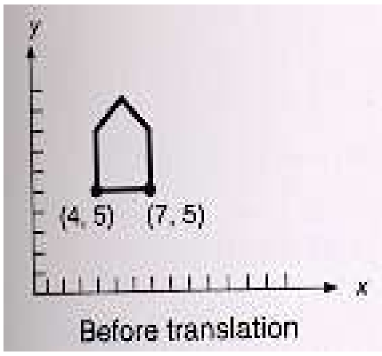
- Moves points to new locations by adding translation amounts to the coordinates of the points

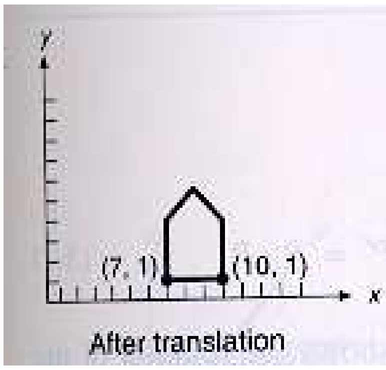


$$x' = x + dx, \quad y' = y + dy \text{ or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\underline{P' = P + T}$$

- To translate an object, translate every point of the object by the same amount

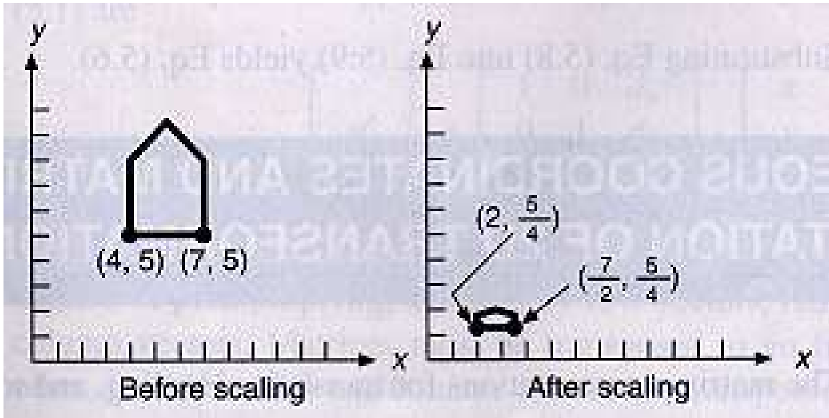




(translate only the endpoints of line segments - redrawing is required)

- **Scaling**

  - Changes the size of the object by multiplying the coordinates of the points by scaling factors



$$x' = x\ s_x,\ \ y' = y\ s_y \ \text{ or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},\ \ s_x, s_y > 0$$

$$\underline{P' = S\ P}$$

- Scale factors affect size as following:

  * If $s_x = s_y$ uniform scaling

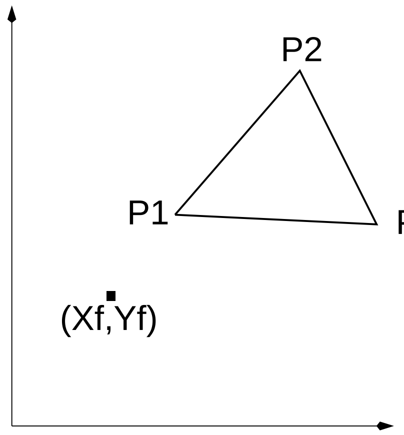
  * If $s_x \neq s_y$ nonuniform scaling

  * If $s_x, s_y < 1$, size is reduced, object moves closer to origin

  * If $s_x, s_y > 1$, size is increased, object moves further from origin

  * If $s_x = s_y = 1$, size does not change

- Control the location of a scaled object by choosing the location of a point (*fixed point*) with respect to which the scaling is performed



$$x' = x_f + (x - x_f)s_x \quad \text{or} \quad x' = xs_x + x_f(1 - s_x)$$

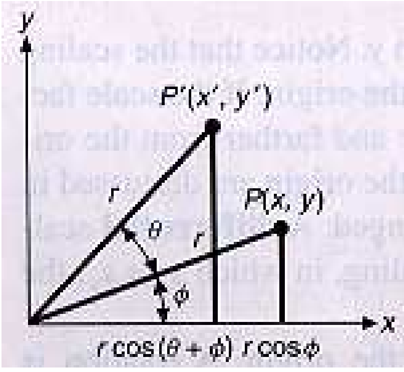
$$y' = y_f + (y - y_f)s_y \quad \text{or} \quad y' = ys_y + y_f(1 - s_y)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_f(1 - s_x) \\ y_f(1 - s_y) \end{bmatrix}$$

$$\underline{P' = S\ P + T_f}$$

- **Rotation**

- Rotates points by an angle θ about origin (θ>0: counterclockwise rotation)



- From ABP triangle:

$cos(\phi) = x/r$ or $x = rcos(\phi)$
$sin(\phi) = y/r$ or $y = rsin(\phi)$

- From ACP' triangle:

$cos(\phi + \theta) = x'/r$ or $x' = rcos(\phi + \theta) = rcos(\phi)cos(\theta) - rsin(\phi)sin(\theta)$
$sin(\phi + \theta) = y'/r$ or $y' = rsin(\phi + \theta) = rcos(\phi)sin(\theta) + rsin(\phi)cos(\theta)$

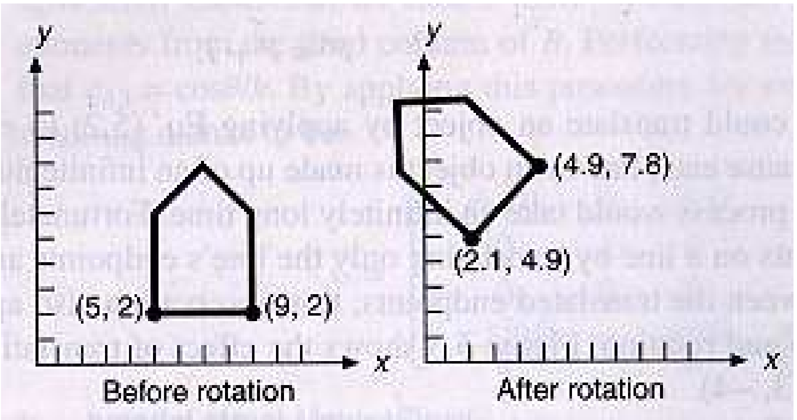
- From the above equations we have:

$$x' = xcos(\theta) - ysin(\theta), \quad y' = xsin(\theta) + ycos(\theta) \text{ or}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

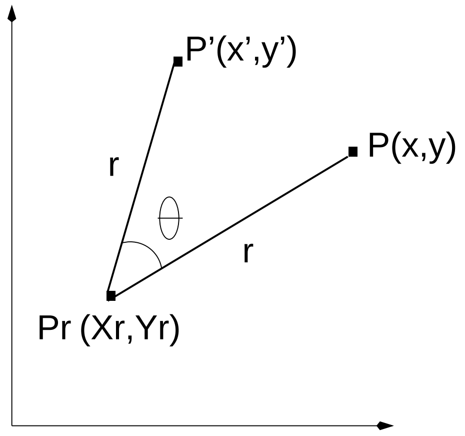
$$\underline{P' = R\ P}$$

- To rotate an object, rotate every point of the object by the same amount



(rotate only the endpoints of line segments - redrawing is required)

- Performing rotation about an arbitrary point



$$x' = x_r + (x - x_r)cos(\theta) - (y - y_r)sin(\theta)$$

$$y' = y_r + (x - x_r)sin(\theta) + (y - y_r)cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\underline{P' = R(P - P_r) + P_r}$$

- **Summary of transformations**

Translation: $P' = P + T$ (addition causes problems !!)

Scale: $P' = S\ P$

Rotation: $P' = R\ P$

- Idea: use homogeneous coordinates to express translation as matrix multiplication

- **Homogeneous coordinates**

- Idea: add a third coordinate: (x, y) --> (x_h, y_h, w)

- Homogenize (x_h, y_h, w):

$$x = \frac{x_h}{w}, \;\; y = \frac{y_h}{w}, \; w \neq 0$$

- In general: (x, y) --> (xw, yw, w) (i.e., x_h=xw, y_h=yw)

- w can assume any value ($w \neq 0$), for example, $w = 1$:

$(x, y) --> (x, y, 1)$ (no division is required when you homogenize !!)

$(x, y) --> (2x, 2y, 2)$ (division is required when you homogenize !!)

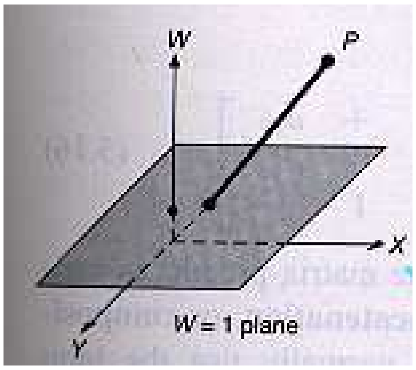
- (x, y) can be represented by an infinite number of points in homogeneous coordinates

If $w = 6$, $(1/3, 1/2) --> (2, 3, 6)$

If $w = 12$, $(1/3, 1/2) --> (4, 6, 12)$

...

- All these points lie on a line in the space of homogeneous coordinates !!



- **Translation using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

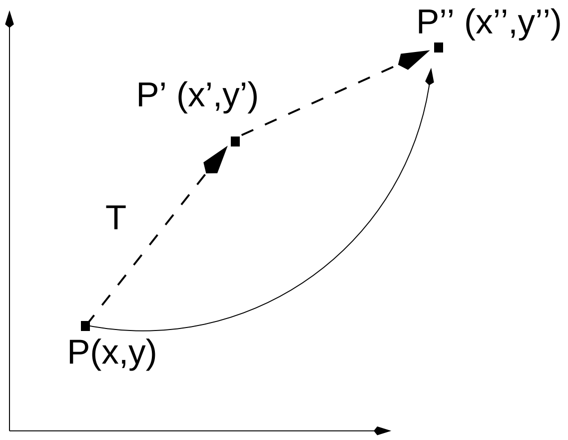
(Verification: $x' = 1x + 0y + 1dx = x + dx$)

(Verification: $y' = 0x + 1y + 1dy = y + dy$)

(Homogenize: divide by 1 !!)

$$\underline{P' = T(dx, dy)\ P}$$

- Successive translations



$$P' = T(dx_1, dy_1)\ P, \quad P'' = T(dx_2, dy_2)\ P'$$

Thus, $P'' = T(dx_2, dy_2)T(dx_1, dy_1)\ P = T(dx_1 + dx_2, dy_1 + dy_2)\ P$

$$\begin{bmatrix} 1 & 0 & dx_2 \\ 0 & 1 & dy_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & dx_1 \\ 0 & 1 & dy_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx_1 + dx_2 \\ 0 & 1 & dy_1 + dy_2 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Scaling using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Verify: $x' = s_x x + 0y + 01 = s_x x$)

(Verify: $y' = 0x + s_y y + 01 = s_y y$)

(homogenize: divide by 1 !!)

$$\underline{P' = S(s_x, s_y)\ P}$$

- Successive scalings

$$P' = S(s_{x_1}, s_{y_1})\ P, \quad P'' = S(s_{x_2}, s_{y_2})\ P'$$

Thus, $P'' = S(s_{x_2}, s_{y_2}) S(s_{x_1}, s_{y_1})\ P = S(s_{x_1} s_{x_2}, s_{y_1} s_{y_2})\ P$

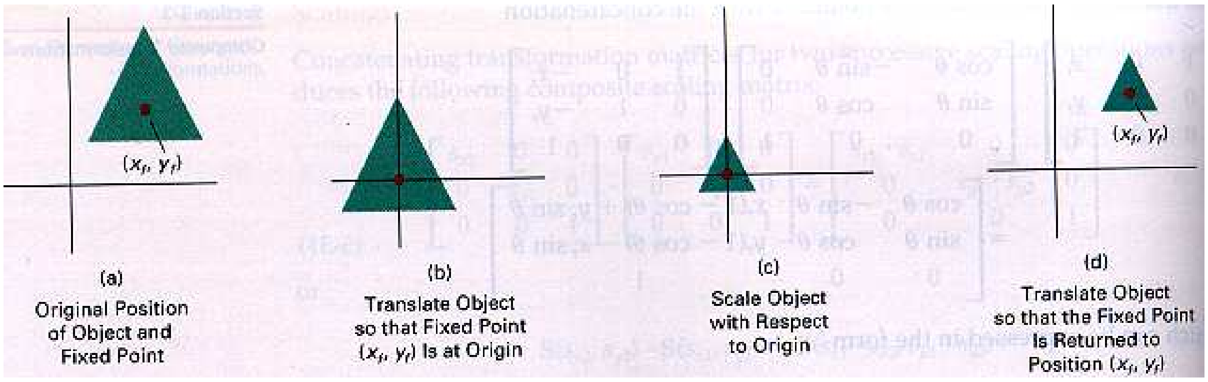
$$\begin{bmatrix} s_{x_2} & 0 & 0 \\ 0 & s_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x_1} & 0 & 0 \\ 0 & s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x_2} s_{x_1} & 0 & 0 \\ 0 & s_{y_2} s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Scaling about a fixed point

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Decomposing the above transformation:

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix}$$



- **Rotation using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underline{P' = R(\theta)\ P}$$

- Successive rotations

$$P' = R(\theta_1)\ P, \quad P'' = R(\theta_2)\ P'$$

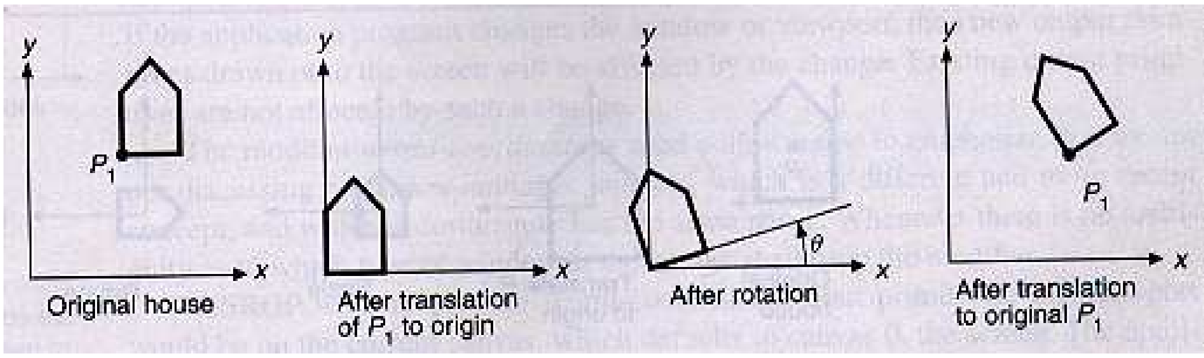
$$\text{Thus, } P'' = R(\theta_1)R(\theta_2)\ P = R(\theta_1 + \theta_2)\ P$$

- Rotation about an arbitrary point

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & x_r(1 - cos(\theta)) + y_r sin(\theta) \\ sin(\theta) & cos(\theta) & y_r(1 - cos(\theta)) - x_r sin(\theta) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Decomposing the above transformation:

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$



- **Composition of a series of transformations**

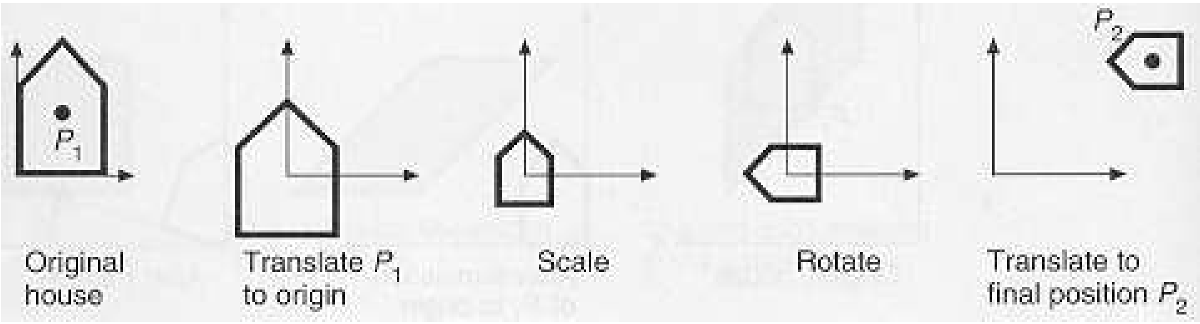
  - The transformation matrices of a series of transformations can be concatenated into a single transformation matrix

  - Example

    * Translate P_1 to origin

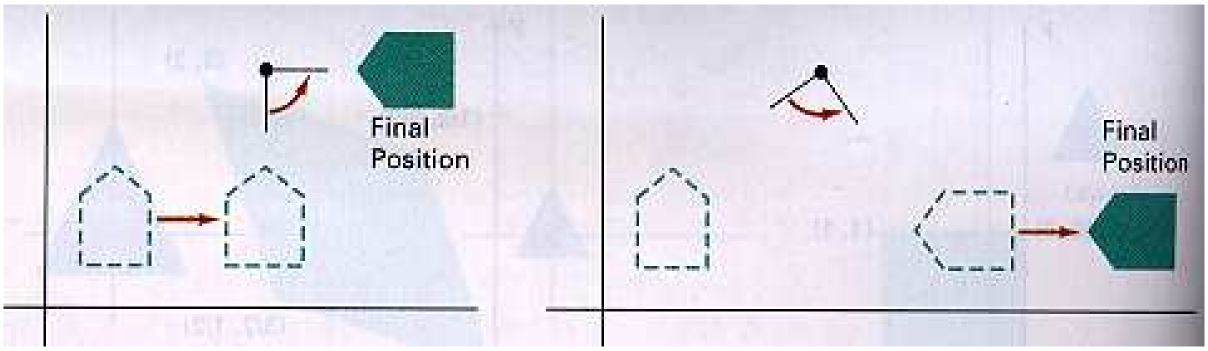
    * Perform scaling and rotation

    * Translate to P_2



$$M = T(x_2, y_2)R(\theta)S(s_x, s_y)T(-x_1, -y_1)$$

  - It is important to reserve the order in which a sequence of transformations is performed !!



- **General form of transformation matrix**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Representing a sequence of transformations as a single transformation matrix is more efficient

$$x' = a_{11}x + a_{12}y + a_{13}$$

$$y' = a_{21}x + a_{22}y + a_{23}$$

only 4 multiplications and 4 additions

- **Rigid-body transformations**

- Involve only translations and rotations

- Preserve angles and lengths

- General form of a rigid-body transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Properties

  * The upper 2x2 submatrix is ortonormal

    $u_1 = (r_{11}, r_{12}),\ u_2 = (r_{21}, r_{22})$

    $u_1 . u_1 = \|u_1\|^2 = {r_{11}}^2 + {r_{12}}^2 = 1$

    $u_2 . u_2 = \|u_2\|^2 = {r_{21}}^2 + {r_{22}}^2 = 1$

    $u_1 . u_2 = r_{11}r_{21} + r_{12}r_{22} = 0$

- Example:

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_1 . u_1 = cos(\theta)^2 + sin(-\theta)^2 = 1$$

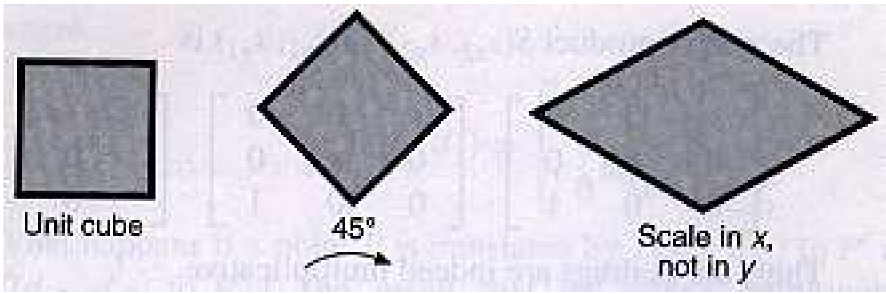
$$u_2 . u_2 = cos(\theta)^2 + sin(\theta)^2 = 1$$

$$u_1 . u_2 = cos(\theta)sin(\theta) - sin(\theta)cos(\theta) = 0$$

- **Affine transformations**

  - Involve translations, rotations, scale, and shear

  - Preserve parallelism of lines but not lengths and angles



- **Shear transformations**

  - Shear along the x-axis:

$$x' = x + ay,\ y' = y$$

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Shear along the y-axis:

$$x' = x,\ y' = bx + y$$

$$SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$