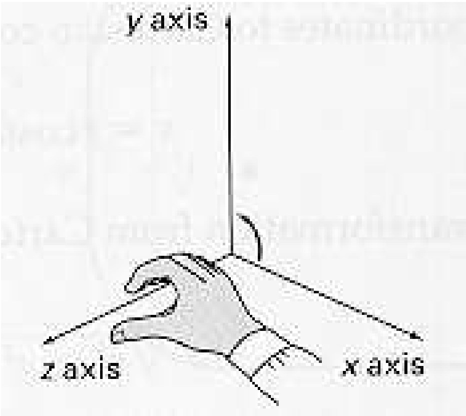
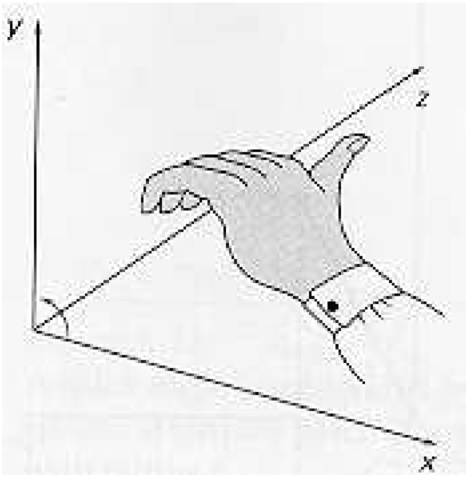
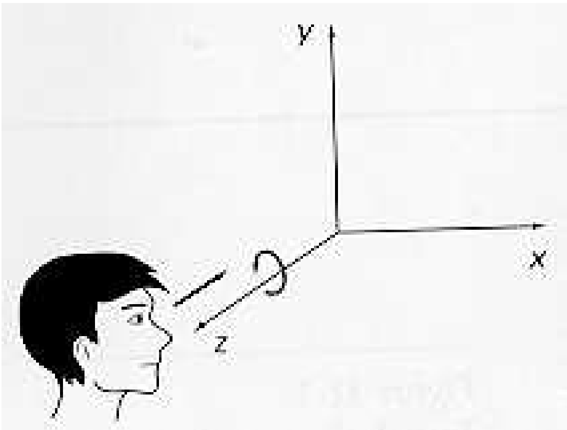
Review of Vectors (cont.)

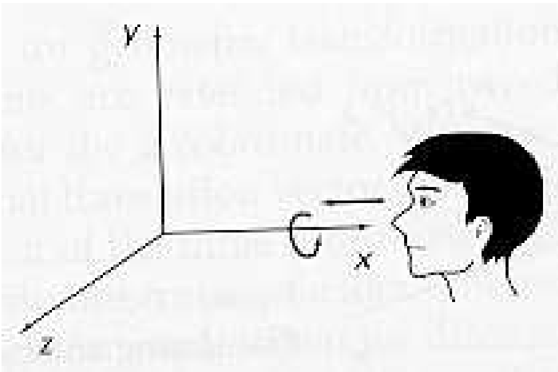
- **Right-handed** versus **left-handed** systems

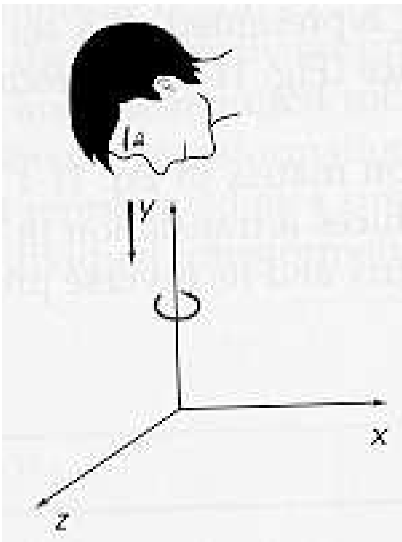




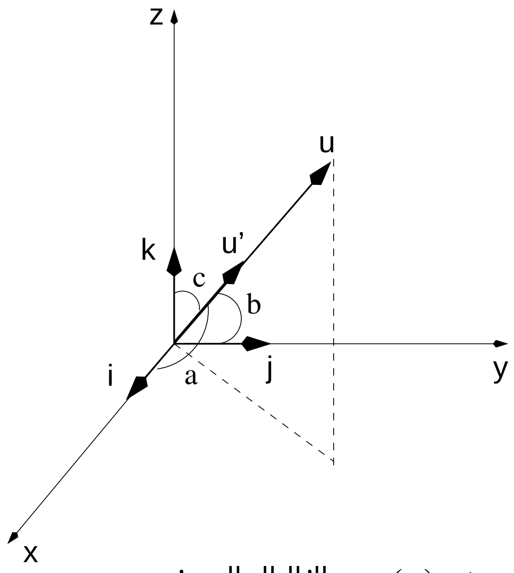
- **Positive rotation angles for right-handed systems**







- **Direction angles and direction cosines**



$$u\ .\ i = \|u\|\ \|i\|\ cos(a) \Rightarrow cos(a) = \frac{u\ .\ i}{\|u\|} = \frac{u_x}{\|u\|}$$

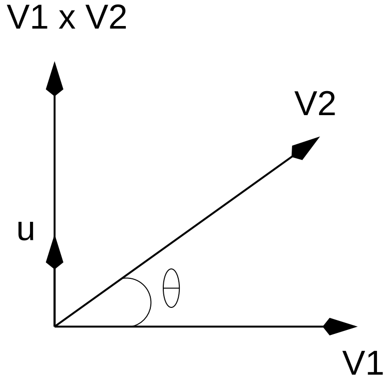
$$u\ .\ j = \|u\|\ \|j\|\ cos(b) \Rightarrow cos(b) = \frac{u\ .\ j}{\|u\|} = \frac{u_y}{\|u\|}$$

$$u\ .\ k = \|u\|\ \|k\|\ cos(c) \Rightarrow cos(c) = \frac{u\ .\ k}{\|u\|} = \frac{u_z}{\|u\|}$$

- $u' = \frac{u}{\|u\|} = (\frac{u_x}{\|u\|}, \frac{u_y}{\|u\|}, \frac{u_z}{\|u\|}) = (cos(a), cos(b), cos(c))$

- Property: $\cos^2(a) + \cos^2(b) + \cos^2(c) = 1$

- **Cross product**



$$v_1\ x\ v_2 = u\ \|v_1\|\ \|v_2\|\ sin(\theta),\ 0 \le \theta \le \pi$$

($v_1\ x\ v_2$ is perpendicular to both v_1 and v_2)

$$u = \frac{v_1 \ x \ v_2}{\|v_1 \ x \ v_2\|} \text{ (unit vector in the direction of } v_1 x v_2)$$

$$v_1 \ x \ v_2 = \begin{vmatrix} i & j & k \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{vmatrix} =$$

$$=(v_{1y}v_{2z} - v_{1z}v_{2y})i + (v_{1z}v_{2x} - v_{1x}v_{2z})j + (v_{1x}v_{2y} - v_{1y}v_{2x})k$$

- **Homogeneous coordinates of a 3D point**

- Idea: add a third coordinate: (x, y, z) --> (x_h, y_h, z_h, w)

- Homogenize (x_h, y_h, z_h, w):

$$x = \frac{x_h}{w}, \ y = \frac{y_h}{w}, z = \frac{z_h}{w}, w \neq 0$$

- In general: (x, y, z) --> (xw, yw, zw, w) (i.e., x_h=xw, y_h=yw, z_h=zw)

- w can assume any value ($w \neq 0$), for example, $w = 1$:

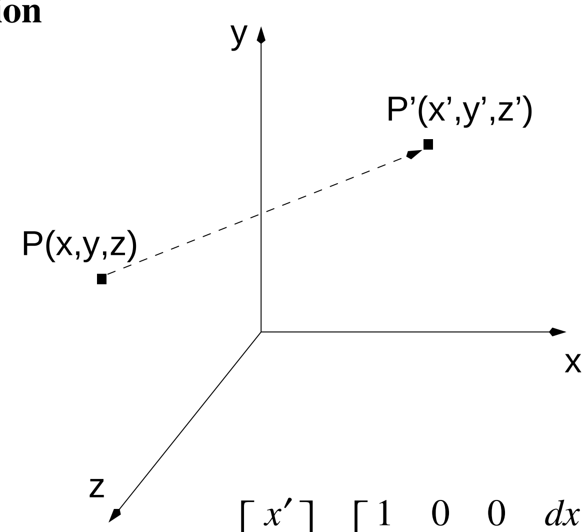
$(x, y, z) --> (x, y, z, 1)$ (no division is required when you homogenize !!)

$(x, y, z) --> (2x, 2y, 2z, 2)$ (division is required when you homogenize !!)

- Each point (x, y, z) correponds to a line in the 4D-space of homogeneous coordinates

3D Geometrical Transformations

- **Translation**



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(Verification: $x' = 1x + 0y + 0z + 1dx = x + dx$)

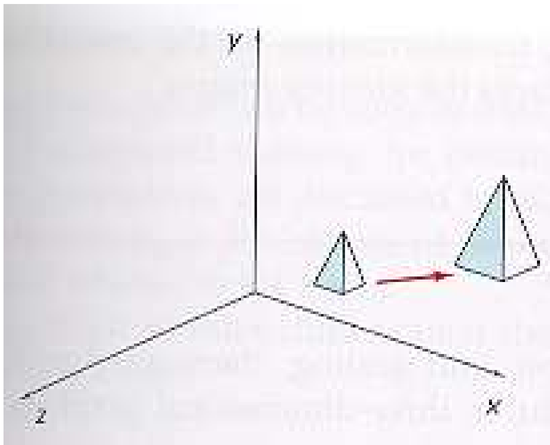
(Verification: $y' = 0x + 1y + 0z + 1dy = y + dy$)

(Verification: $z' = 0x + 0y + 1z + 1dz = z + dz$)

(Homogenize: divide by 1 !!)

$\underline{P' = T(dx, dy, dz)\ P}$

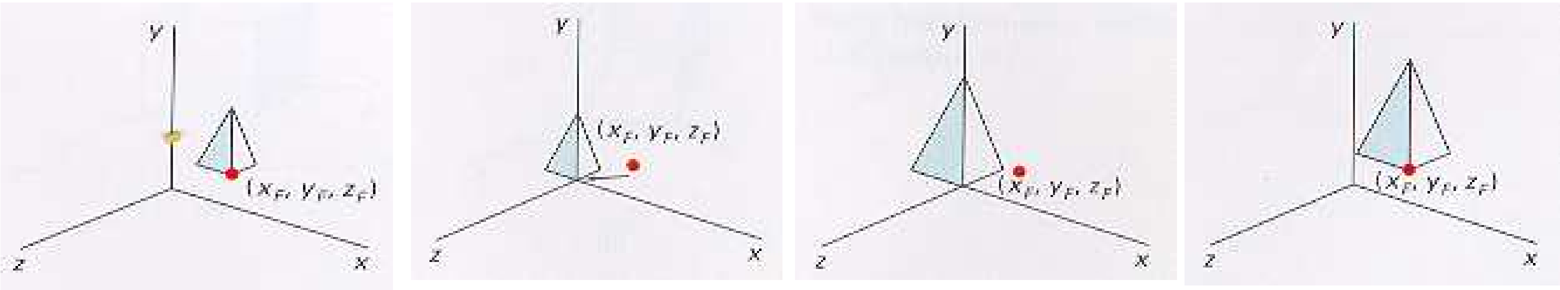
- **Scale**



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = S(s_x, s_y, s_z)\ P}$$

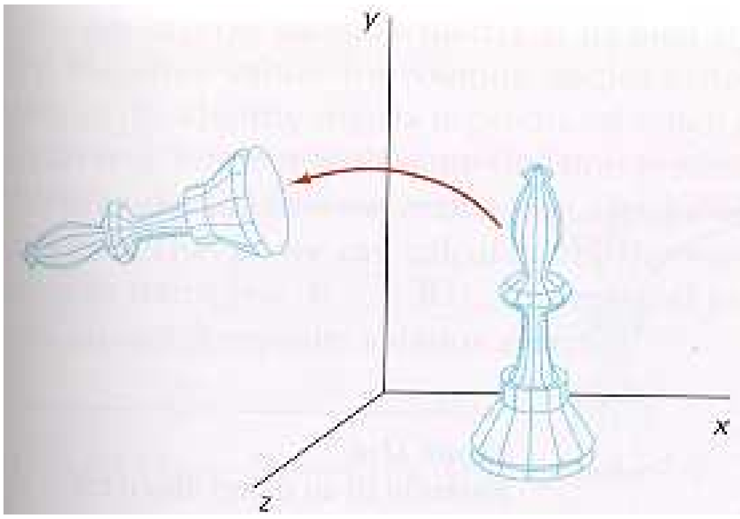
- Scaling about a fixed point



$$\begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Rotation**

- <u>Rotation about the z-axis</u>



$$x' = x\cos(\theta) - y\sin(\theta)$$

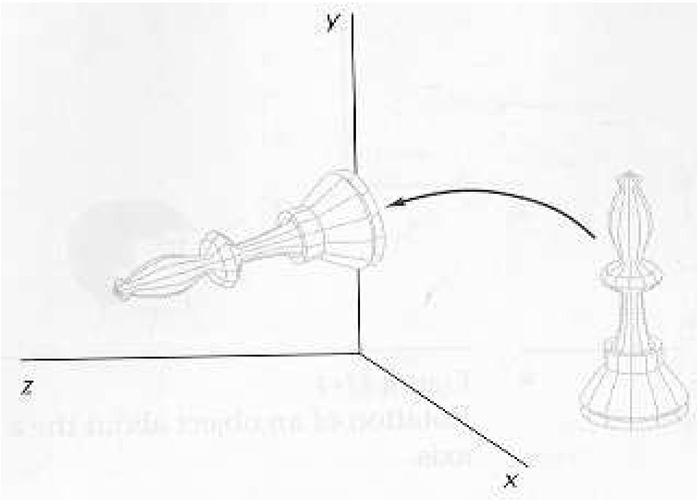
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_z(\theta)\ P}$$

- <u>Rotation about the x-axis</u>



$$x' = x$$

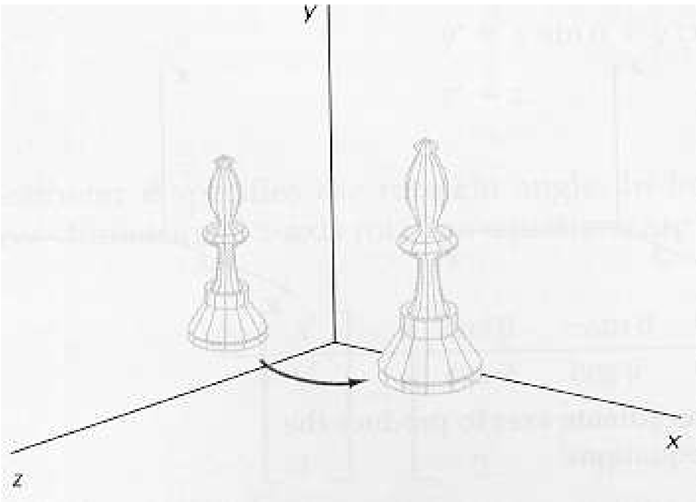
$$y' = y\cos(\theta) - z\sin(\theta)$$

$$z' = y\sin(\theta) + z\cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_x(\theta)\ P}$$

- Rotation about the y-axis



$$x' = z\sin(\theta) + x\cos(\theta)$$

$$y' = y$$

$$z' = z\cos(\theta) - x\sin(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_y(\theta)\ P}$$

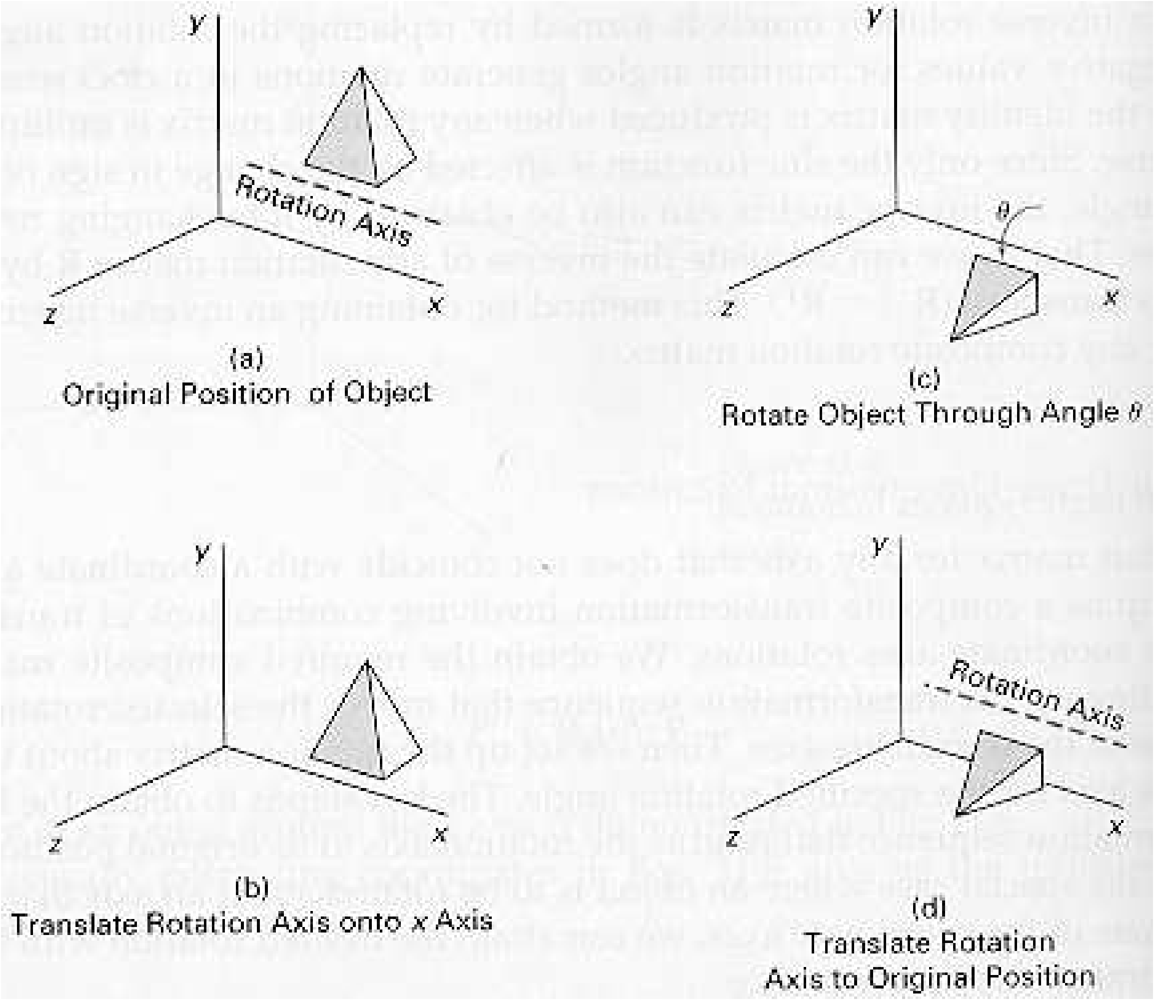
- **Rotation about a general axis**

- Case 1: the axis is parallel to one of the coordinate axes

1. Translate the object so that the rotation axis coincides with the parallel coordinate axis

2. Perform the rotation about that axis

3. Apply the inverse translation to bring the rotation axis to its original location



$$P' = T(-dx, -dy, -dz)R_x(\theta)T(dx, dy, dz)P$$

- Case 2: the axis is not parallel to any one of the coordinate axes

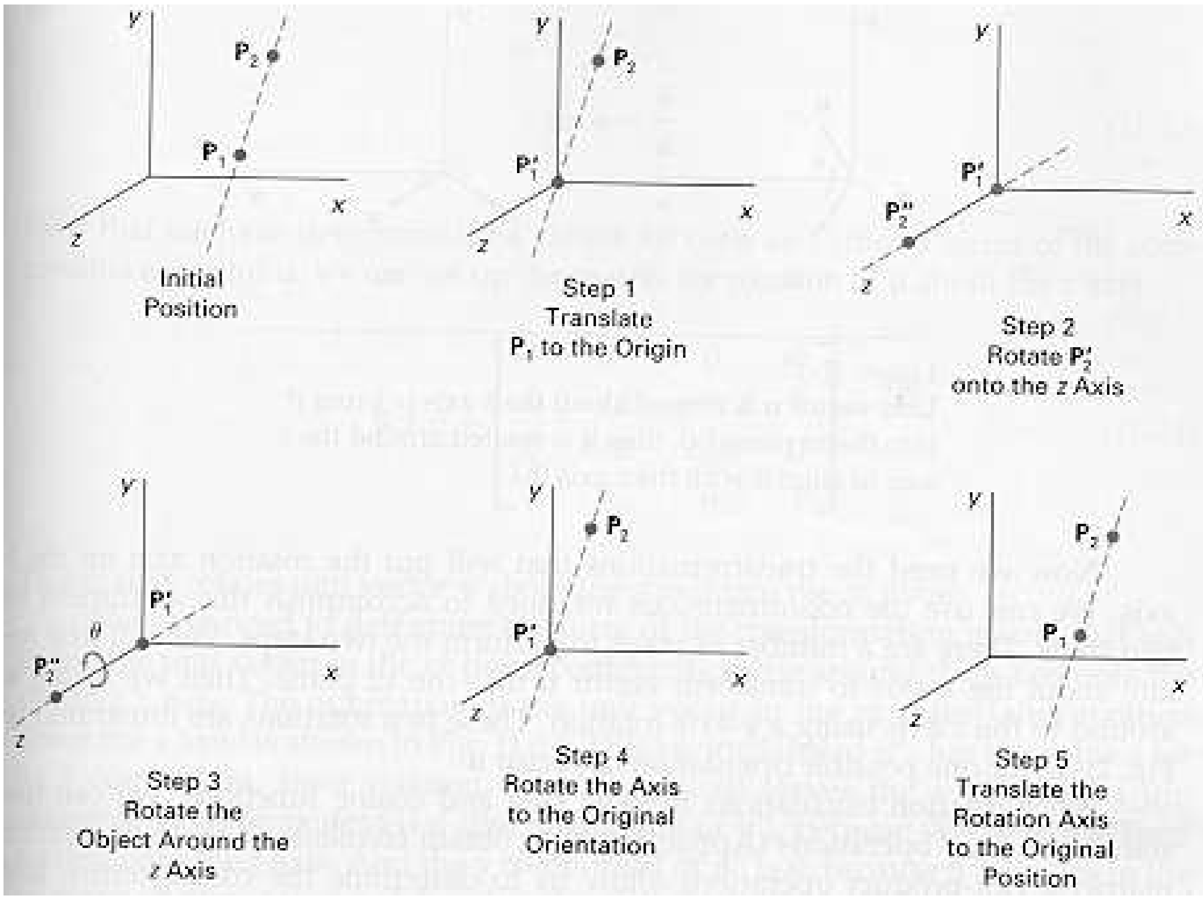
1. Translate the object so that the rotation axis passes through the origin

2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes

3. Perform the specified rotation about that coordinate axis

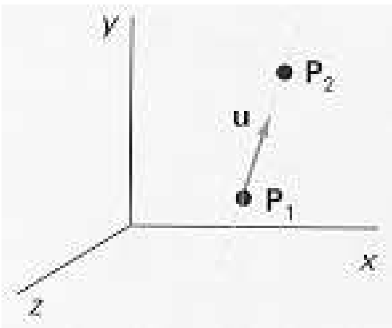
4. Apply the inverse rotations to bring the rotation axis to its original orientation

5. Apply the inverse translation to bring the rotation axis to its original location



$$P' = T(-dx, -dy, -dz)R_x^{-1}(\theta_1)R_y^{-1}(\theta_2)R_z(\phi)R_y(\theta_1)R_x(\theta_2)T(dx, dy, dz)P$$

- Computing the transformation matrices

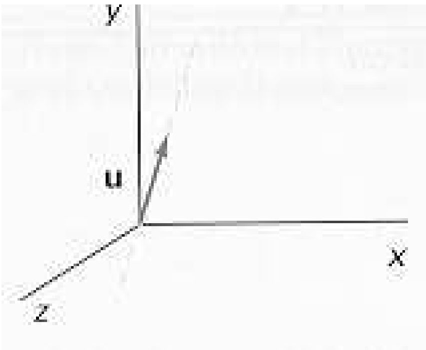


a. Pick two points P_1 and P_2 on the rotation axis

b. $V = P_2 - P_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

c. Define the unit vector $u = \dfrac{v}{\|v\|} = (\dfrac{x_2 - x_1}{\|v\|}, \dfrac{y_2 - y_1}{\|v\|}, \dfrac{z_2 - z_1}{\|v\|})$

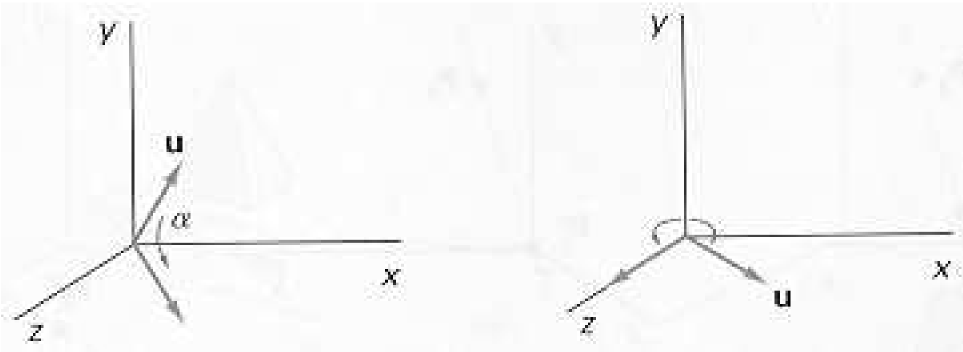
1. Computing T



Translate P_1 to (0,0,0)

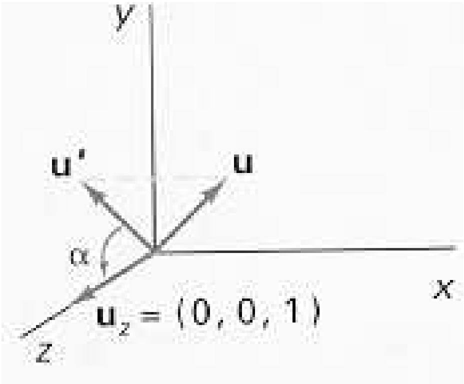
$$\begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Computing $R_y(\theta_1)$, $R_x(\theta_2)$



$$R_x(\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_2) & -sin(\theta_2) & 0 \\ 0 & sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; R_y(\theta_1) = \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta_1) & 0 & cos(\theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Computing $cos(\theta_2)$, $sin(\theta_2)$



a. Consider u', the projection of u on the yz plane
(if $u = (a, b, c)$, then $u' = (0, b, c)$)

b. Consider the dot product between u' and $u_z = (0, 0, 1)$

$$u' \,.\, u_z = \|u'\| \, \|u_z\| \, cos(\theta_2) \text{ or}$$

$$cos(\theta_2) = \frac{u' \,.\, u_z}{\|u'\|} = \frac{00 + b0 + c1}{\sqrt{b^2 + c^2}} = \frac{c}{d}$$

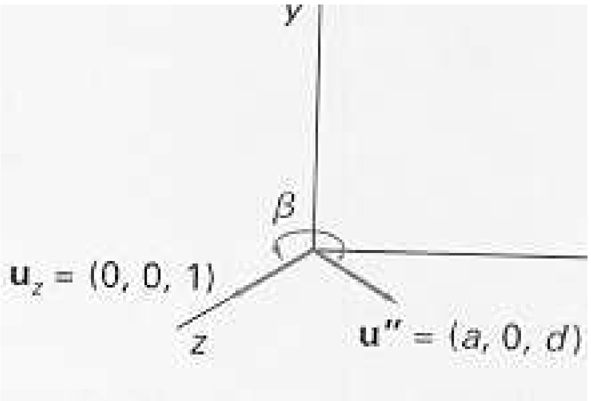
c. Consider the cross product between u' and u_z

$$u' \; x \; u_z = u_x \, \|u'\| \, \|u_z\| \, sin(\theta_2) \text{ or}$$

$$sin(\theta_2) = \frac{u' \; x \; u_z}{u_x \|u'\|} = \frac{b}{d}$$

$$R_x(\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Computing $cos(\theta_1)$, $sin(\theta_1)$



a. Consider $u'' = R_x(\theta_2)u = (a, 0, d)$

b. Consider the dot product between u'' and $u_z = (0, 0, 1)$

$$u'' \,.\, u_z = \|u''\| \, \|u_z\| \; cos(\theta_1) \text{ or}$$

$$cos(\theta) = \frac{u'' \,.\, u_z}{\|u''\|} = \frac{0a + 00 + 1d}{\sqrt{a^2 + d^2}} = d \quad (\text{since } d = \sqrt{b^2 + c^2})$$

c. Consider the cross product between u'' and u_z

$$u'' \; x \; u_z = - \, u_y \|u''\| \, \|u_z\| \; sin(\theta) \text{ or}$$

$$sin(\theta) = - \, \frac{u'' \; x \; u_z}{u_y \|u''\|} = - \, \frac{(-a)}{1} = a$$

$$R_y(\theta_1) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Computing $R_z(cos(\phi))$

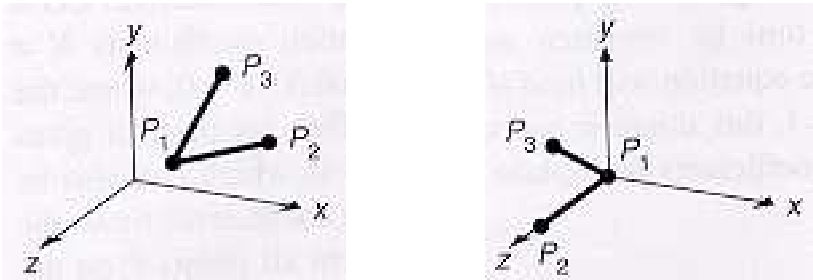
$$R_z(\phi) = \begin{bmatrix} cos(\phi) & -sin(\phi) & 0 & 0 \\ sin(\phi) & cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Composition of 3D transformations**

- Composite transformations can be formed by multiplying the matrix representations of the individual operations in the transformation sequence

- Concatenation is carried out from right to left, where the rightmost matrix is the first transformation matrix to be applied and the leftmost matrix is the last transformation

- Example

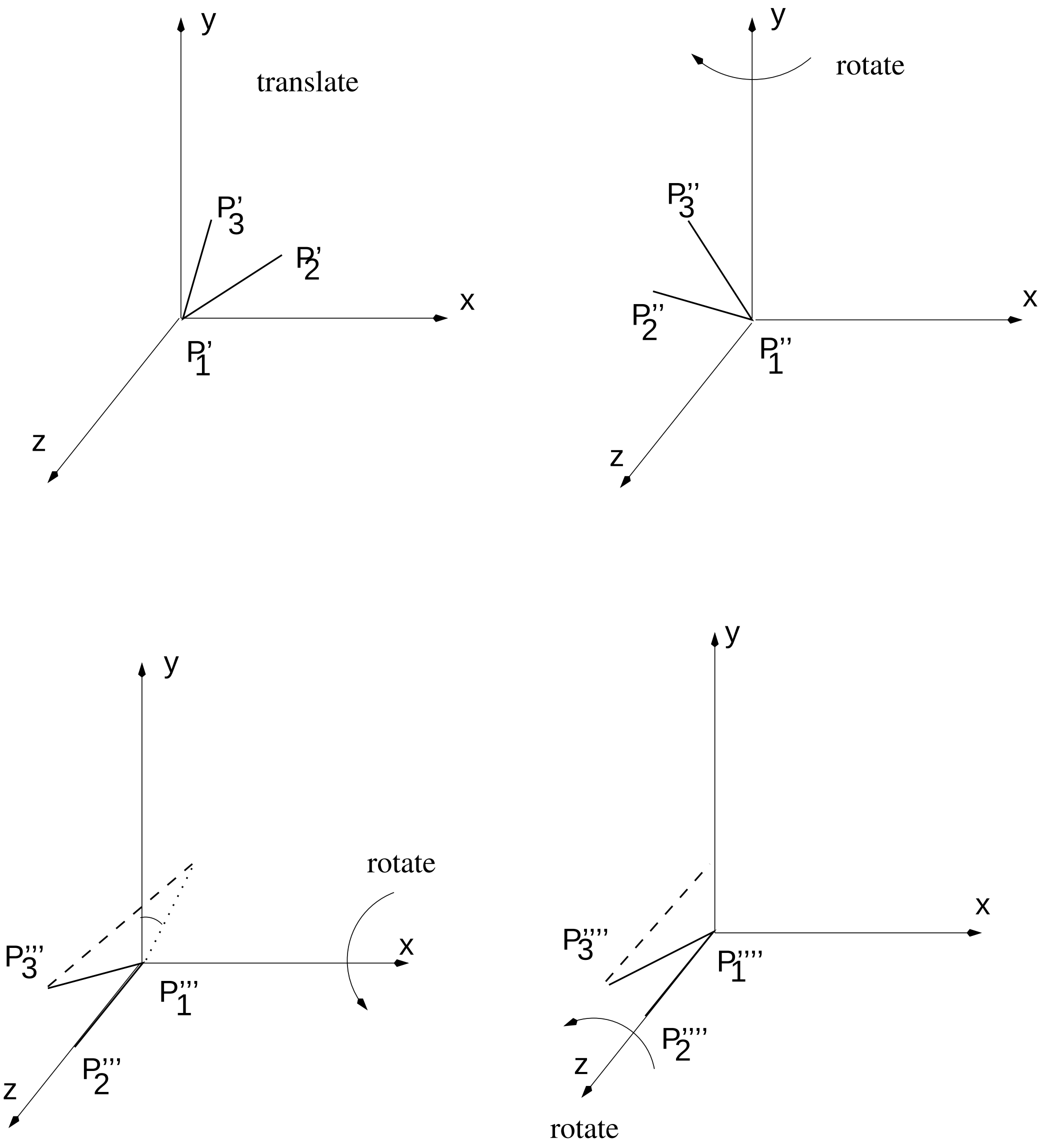


1. Translate P_1 to origin

2. Rotate about the y-axis such that P_1P_2 lies in the yz plane

3. Rotate about the x-axis such that P_1P_2 lies on the z-axis

4. Rotate about the z-axis such that P_1P_3 lies in the yz plane



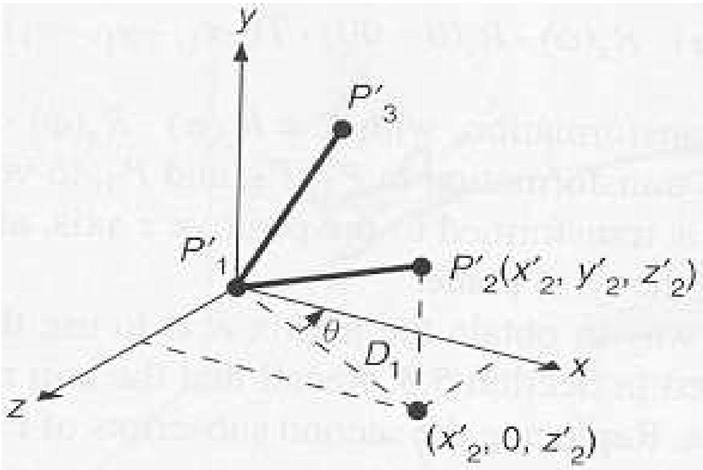
- Computing the corresponding transformation matrices

1. Translate P_1 to origin

$$T(-x_1, -y_1, -z_1) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P'_1 = TP_1,\ P'_1 = TP_3,\ P'_3 = TP_3$$

2. Rotate about the y-axis such that P_1P_2 lies in the yz plane



* What is the angle of rotation ? $-(90-\theta)=\theta-90$

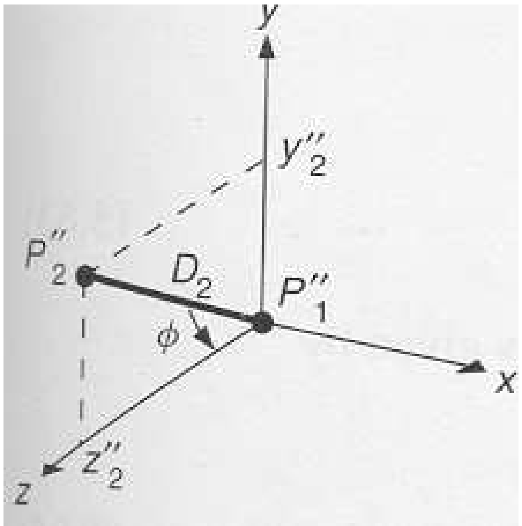
$$R_y(\theta-90)=\begin{bmatrix} cos(\theta-90) & 0 & sin(\theta-90) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta-90) & 0 & cos(\theta-90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} sin(\theta) & 0 & -cos(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Compute $cos(\theta)$, $sin(\theta)$

$$cos(\theta)=\frac{x_2^{'}}{D_1},\quad sin(\theta)=\frac{z_2^{'}}{D_1},\quad (D_1=\sqrt{(x_2^{'})^2+(z_2^{'})^2})$$

$$P_2^{''}=R_y(\theta-90)P_2^{'},\ P_3^{''}=R_y(\theta-90)P_3^{'}$$

3. Rotate about the x-axis such that P_1P_2 lies on the z-axis



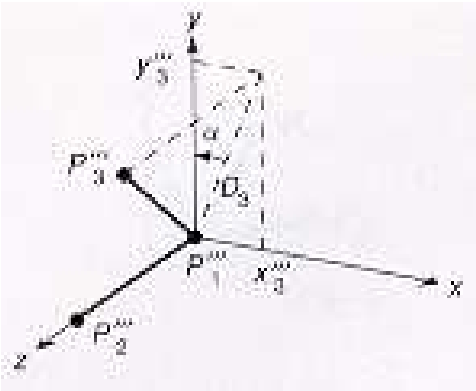
$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) & 0 \\ 0 & sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Compute $cos(\phi)$, $sin(\phi)$

$$cos(\phi) = \frac{z_2^{''}}{D_2}, \quad sin(\phi) = \frac{y_2^{''}}{D_2}, \quad (D_2 = \sqrt{(y_2^{''})^2 + (z_2^{''})^2})$$

$$P_2^{'''} = R_x(\phi)P_2^{''},\ P_3^{'''} = R_x(\phi)P_3^{''}$$

4. Rotate about the z-axis so that P_1P_3 lies in the yz plane



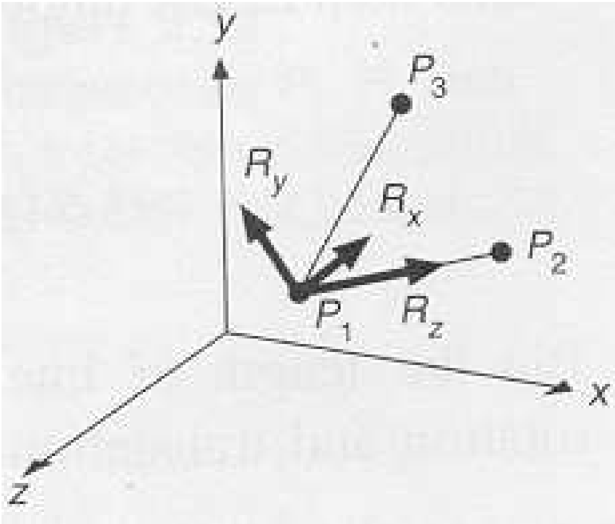
$$R_z(a) = \begin{bmatrix} cos(a) & -sin(a) & 0 & 0 \\ sin(a) & cos(a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Compute $cos(a)$, $sin(a)$

$$cos(a) = \frac{y_3^{'''}}{D_3}, \quad sin(a) = \frac{x_3^{'''}}{D_2}, \quad (D_3 = \sqrt{(x_3^{'''})^2 + (y_3^{'''})^2})$$

Composite matrix: $R_z(a)R_x(\phi)R_y(\theta - 90)T(-x_1, -y_1, -z_1) = RT(-x_1, -y_1, -z_1)$

- An alternative way to obtain the composite rotation matrix R



* Assume a local coordinate system

$$r_z = \frac{P_1P_2}{\|P_1P_2\|}$$

(unit vector along P_1P_2)

$$r_x = \frac{(P_1P_3)x(P_1P_2)}{\|(P_1P_3)x(P_1P_2)\|}$$

(unit vector, perpendicular to the plane defined by P_1P_3 and P_1P_2)

$$r_y = r_z \ x \ r_x$$

(unit vector, perpendicular to the plane defined by r_z and r_x)

* Idea: find a mapping that will map $r_z \rightarrow u_z$, $r_y \rightarrow u_x$, and $r_y \rightarrow u_y$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{z_x} \\ r_{z_y} \\ r_{z_z} \\ 1 \end{bmatrix} \quad (r_z \rightarrow u_z) \tag{1}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{x_x} \\ r_{x_y} \\ r_{x_z} \\ 1 \end{bmatrix} \quad (r_x \rightarrow u_x) \tag{2}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{y_x} \\ r_{y_y} \\ r_{y_z} \\ 1 \end{bmatrix} \quad (r_y \rightarrow u_y) \qquad (3)$$

From (1): $a_3^T r_z = 1$ or $a_3 = r_z$

From (2): $a_1^T r_x = 1$ or $a_1 = r_x$

From (3): $a_2^T r_y = 1$ or $a_2 = r_y$

$$\text{Thus, } R = \begin{bmatrix} & r_x^T & & 0 \\ & r_y^T & & 0 \\ & r_z^T & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

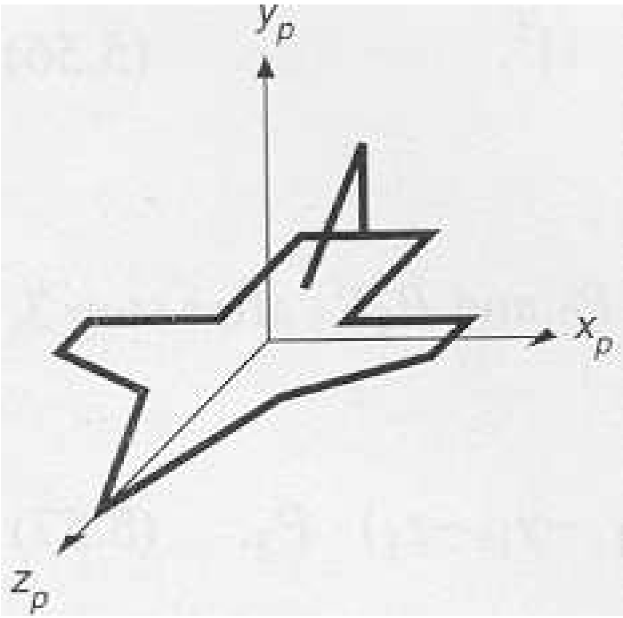
- Verify

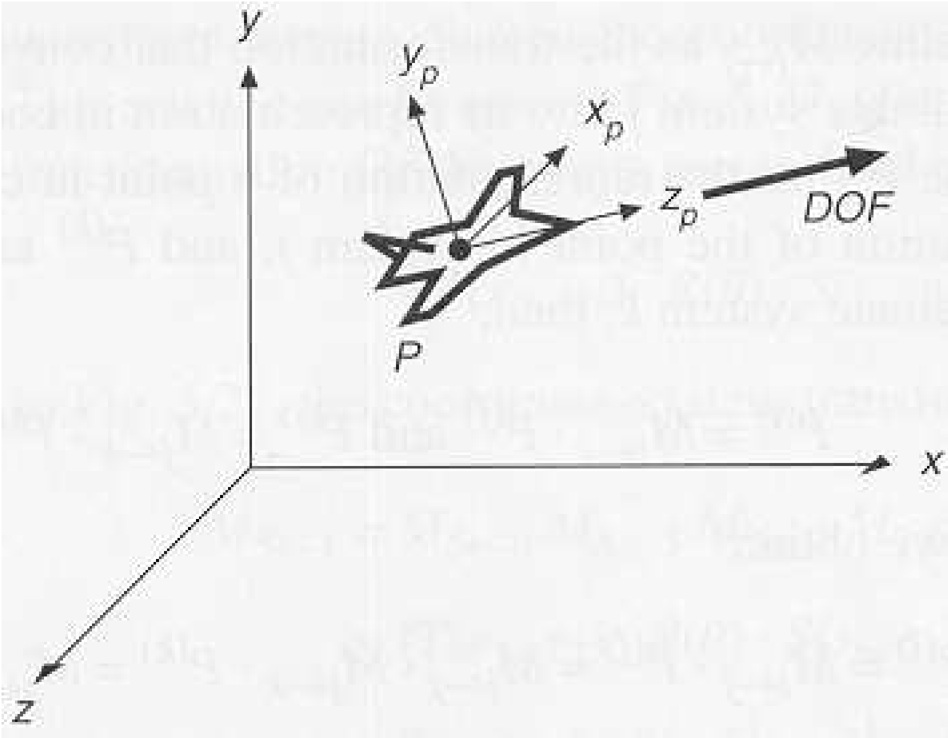
$$R \begin{bmatrix} r_x \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R \begin{bmatrix} r_y \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$R \begin{bmatrix} r_z \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

- Example





$$u_z' = \frac{DOF}{\|DOF\|}, \quad u_x' = \frac{u_x \ x \ u_z'}{\|u_x \ x \ u_z'\|}, \quad u_y' = u_z' \ x \ u_x'$$

* Suppose that we want to map $u_z' \rightarrow u_z$, $u_x' \rightarrow u_x$, and $u_y' \rightarrow u_y$

$$R = \begin{bmatrix} & u_x'^T & & 0 \\ & u_y'^T & & 0 \\ & u_z'^T & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Actually, we want to map $u_z \rightarrow u_z'$, $u_x \rightarrow u_x'$, and $u_y \rightarrow u_y'$

$$R^T = \begin{bmatrix} & & & 0 \\ u_x' & u_y' & u_z' & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$