Review on Matrices

• Matrix multiplication

$$\begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & b_{1p} \\ b_{21} & b_{22} & . & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{q1} & b_{q2} & . & b_{qp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & c_{1p} \\ c_{21} & c_{22} & . & c_{2p} \\ \cdots & \cdots & c_{ij} & \cdots \\ c_{m1} & c_{m2} & . & c_{mp} \end{bmatrix}$$

# of columns of matrix A = # of rows of matrix B

(if A is m x n and B is q x p, then C will be m x p (assume n=q))

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

• Properties of matrix multiplication

$$A(B + C) = AB + AC$$

$$AB \neq BA$$

$$AI = IA = A, \text{ where } I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & . & 1 \end{bmatrix}$$

• Matrix transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix}, A^T = \begin{bmatrix} a_{11} & a_{21} & . & a_{m1} \\ a_{12} & a_{22} & . & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & . & a_{mn} \end{bmatrix}$$

Property: $(AB)^T = B^T A^T$

• **Determinants** (square matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$det(A) = \sum_{j=1}^{m} (-1)^{j+k} a_{jk} det(A_{jk}), \text{ for any } k\text{: } 1 \le k \le m$$

Property: $det(AB) = det(A)det(B)$

In general: $det(A + B) \neq det(A) + det(B)$

• **Matrix inverse**

- The inverse A^{-1} of matrix A has the property: $AA^{-1}=A^{-1}A=I$

- A^{-1} exists only if $det(A) \neq 0$

- $(AB)^{-1}=B^{-1}A^{-1}$

- $(A^T)^{-1} = (A^{-1})^T$

• **Trace of a matrix** (square matrix)

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

$$tr(A^T) = tr(A)$$

$$tr(A \pm B) = tr(A) \pm tr(B)$$

$$tr(AB) = tr(BA)$$

• Rank of a matrix

- It is equal to the dimension of the largest square submatrix of A that has a non-zero determinant.

- Alternatively, it is the maximum number of linearly independent columns or rows of A.

- if A is mxn, $rank(A) \leq \min m, n$

- If A is nxn, $rank(A) = n$ iff A is nonsingular.

- $rank(A) < n$ iff A is singular

• Dot product

$$v = (v_x, v_y),\ w = (w_x, w_y)$$

$$v.\, w = v_x w_x + v_y w_y \text{ or}$$

$$v.\, w = [v_x \ \ v_y] \begin{bmatrix} w_x \\ w_y \end{bmatrix} = v^T w$$

• Orthogonal/orthonormal matrices

- Consider the vectors formed by the rows (or columns) of matrix A:

$$\begin{aligned} u_1 &= [a_{11} \ a_{12} \cdots a_{1n}] \\ u_2 &= [a_{21} \ a_{22} \cdots a_{2n}] \\ &\ldots \\ u_m &= [a_{m1} \ a_{m2} \cdots a_{mn}] \end{aligned}$$

<u>Property (1):</u> $u_k . u_k = 1$ or $\|u_k\| = 1$, for every k

<u>Property (2):</u> $u_j . u_k = 0$, for every $j \neq k$ (u_j is perpendicular to u_k)

<u>A is orthonormal</u> if both (1) and (2) are satisfied

<u>A is orthogonal</u> if only (2) is satisfied