# An Ordering Algorithm for Pattern Presentation in Fuzzy ARTMAP That Tends to Improve Generalization Performance

Issam Dagher, Michael Georgiopoulos, *Member, IEEE,* Gregory L. Heileman, *Senior Member, IEEE,* and George Bebis, *Member, IEEE*

*Abstract*—**In this paper we introduce a procedure, based on the max–min clustering method, that identifies a fixed order of training pattern presentation for fuzzy adaptive resonance theory mapping (ARTMAP). This procedure is referred to as the ordering algorithm, and the combination of this procedure with fuzzy ARTMAP is referred to as ordered fuzzy ARTMAP. Experimental results demonstrate that ordered fuzzy ARTMAP exhibits a generalization performance that is better than the average generalization performance of fuzzy ARTMAP, and in certain cases as good as, or better than the best fuzzy ARTMAP generalization performance. We also calculate the number of operations required by the ordering algorithm and compare it to the number of operations required by the training phase of fuzzy ARTMAP. We show that, under mild assumptions, the number of operations required by the ordering algorithm is a fraction of the number of operations required by fuzzy ARTMAP.**

*Index Terms*— **Fuzzy ARTMAP, generalization, learning, max–min clustering.**

## I. Introduction

PATTERN classification is a key element in many engineering applications. For example, sonar, radar, seismic, and diagnostic applications all require the ability to accurately classify data. In addition, control, tracking, and prediction systems will often use classifiers to determine input–output relationships. Simpson has identified a number of desirable properties that a pattern classifier should possess [1]. These properties can be summarized as follows: A successful pattern classifier should be able to 1) learn the required task quickly; 2) learn new data without having to retrain with old data (on-line adaptation); 3) solve nonlinearly separable problems; 4) provide the capability for soft and hard decisions regarding the degree of membership of the data within each class; 5) offer explanations of how the data are classified, and why the data are classified as such; 6) exhibit performance that is independent of parameter tuning; 7) function without knowledge of the distributions of the data in each class; and 8) for overlapping pattern classes, create regions in the space of the input parameters that exhibit the least possible overlap.

Manuscript received December 22, 1997; revised February 18, 1998 and October 29, 1998.

I. Dagher and M. Georgiopoulos are with the Department of Electrical and Computer Engineering, University of Central Florida, Orlando, FL 32816 USA.

G. L. Heileman is with the Department of Electrical and Computer Engineering, University of New Mexico, Albuquerque, NM 87131 USA.

G. Bebis is with the Department of Computer Science, University of Nevada-Reno, Reno, NV 89557 USA.

Publisher Item Identifier S 1045-9227(99)05829-4.

A neural-network classifier that satisfies most of the aforementioned properties is fuzzy adaptive resonance theory mapping (fuzzy ARTMAP) [2]. Fuzzy ARTMAP is capable of establishing arbitrary mappings between an analog input space of arbitrary dimensionality and an analog output space of arbitrary dimensionality. Fuzzy ARTMAP is a member of the class of neural-network architectures referred to as *ART-architectures* developed by Carpenter, Grossberg, and colleagues. The ART-architectures are based on the *ART theory* introduced by Grossberg [3].

Fuzzy ARTMAP can operate in *off-line* or *on-line* modes. In the on-line mode, the network must process the data as it becomes available, without storing or reusing it. In the off-line mode, the data can be stored and repeatedly presented to the network. In this paper, we consider the off-line operation of fuzzy ARTMAP in *classification problems* (e.g., recognizing handwritten digits). In particular, we consider one of the major limitations of fuzzy ARTMAP, its dependence on tuning parameters [which is a violation of property 6) above]. It has been documented in the literature that the performance of fuzzy ARTMAP depends on the values of two parameters called the choice and vigilance parameters, and also on the order of pattern presentation for the off-line mode of training. To circumvent the first problem, most fuzzy ARTMAP simulations that have appeared in the literature assume zero values for the choice and vigilance parameters. One of the main reasons for the popularity of this choice is that it tends to minimize the size of the resulting network architecture. This is quite desirable, especially when performance comparisons are made between fuzzy ARTMAP and other neural-network architectures that offer more compact representations of the data, such as multilayer perceptrons [4]. The problem of pattern ordering is not as easy to solve. One way around it is to consider different orders of presentations of the training data, in order to find the one that maximizes the performance of the network. The drawbacks of this approach include the considerable experimentation that is required to find a random order of pattern presentation that achieves a good network performance, and the fact that this is essentially a guessing exercise. In this paper, we preprocess the training data by applying a systematic procedure (based on the Max–Min clustering algorithm [5]), which identifies a fixed order of pattern presentation. We refer to this procedure as the *ordering algorithm*. When the training input patterns are presented to fuzzy ARTMAP according to this fixed order we end up with a trained fuzzy ARTMAP whose generalization performance is better than the average generalization performance of fuzzy ARTMAP, and in certain cases as good as, or better than the best network generalization performance. In the former case we consider the average of a fixed number of experiments corresponding to random orders of training pattern presentations, and in the latter case we consider the best of a fixed number of experiments corresponding to random orders of training pattern presentations. For simplicity, we refer to fuzzy ARTMAP trained with the fixed order of input pattern presentations as *ordered fuzzy ARTMAP*. Ordered fuzzy ARTMAP has the following desirable properties.

1) It achieves good generalization performance without requiring parameter tuning.
2) The sizes of the networks that ordered fuzzy ARTMAP creates are comparable to the sizes of the networks that fuzzy ARTMAP creates when trained using a random order of pattern presentation.
3) Under mild conditions, the computational overhead imposed by the ordering algorithm is small compared to the computations required to perform the training phase of fuzzy ARTMAP for a single random order of pattern presentation.

The organization of this paper is as follows. In Section II, we briefly discuss the fuzzy ARTMAP architecture, including the form of its inputs, training phase, performance phase, and functionality. In Section III, we introduce the ordering algorithm. In Section IV, we illustrate with simple examples the effect of the ordering algorithm on the categories created by fuzzy ARTMAP, and we explain the motivation for choosing this ordering algorithm. In Section V, we experimentally demonstrate the superiority of ordered fuzzy ARTMAP's generalization performance compared to the average fuzzy ARTMAP generalization performance, and in certain cases compared to the best fuzzy ARTMAP generalization performance. Also, in Section V we discuss the computational complexity of the ordering algorithm and compare it to the computational complexity of the training phase of fuzzy ARTMAP. Finally, in Section V we compare the generalization performance of the ordered fuzzy ARTMAP and other classification techniques that have appeared in the literature. Section VI provides a summary of the paper and offers some concluding remarks.

## II. The Fuzzy ARTMAP Neural Network

A detailed description of the fuzzy ARTMAP neural network can be found in [2]; we present only the necessary details here. The fuzzy ARTMAP neural network consists of two fuzzy ART modules, designated $ART_a$ and $ART_b$, as well as an inter-ART module as shown in Fig. 1. Inputs are presented at the $ART_a$ module, while their corresponding outputs are presented at the $ART_b$ module. The inter-ART module includes a MAP field whose purpose is to determine whether the correct input–output mapping has been established. The input pattern,

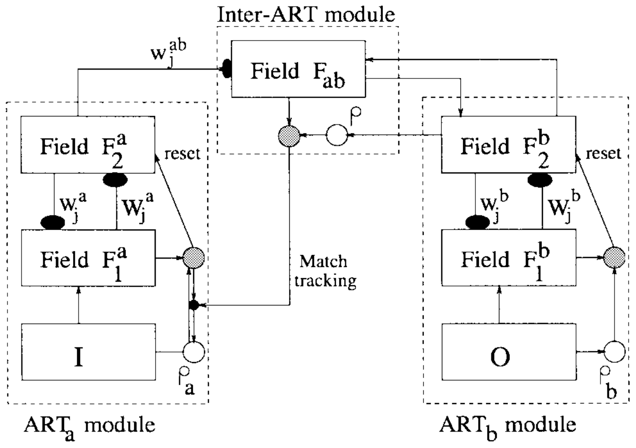


Fig. 1. A block diagram of the fuzzy ARTMAP neural-network architecture.

designated by $\mathbf{I}$ has the form

$$\mathbf{I} = (\mathbf{a}, \mathbf{a}^c) = \left(a_1, \cdots, a_{M_a}, a_1^c, \cdots, a_{M_a}^c\right)$$

where

$$a_i \in [0,1] \quad \text{and} \quad a_i^c = 1 - a_i; \quad 1 \le i \le M_a. \tag{1}$$

The output pattern, designated by $\mathbf{O}$, has the form

$$\mathbf{O} = \left(b_1, \cdots, b_{M_b}\right) \quad \text{where } b_k \in [0,1]; \quad 1 \le k \le M_b. \tag{2}$$

Fuzzy ARTMAP operates in two distinct phases: the *training phase* and the *performance phase*. As mentioned earlier, in this paper we are interested in the off-line operation of fuzzy ARTMAP in classification tasks, where many inputs are mapped to a single, distinct output. The *off-line training phase* of fuzzy ARTMAP works as follows: Given a list of training input–output pairs, such as $\{\mathbf{I}^1, \mathbf{O}^1\}, \cdots \{\mathbf{I}^r, \mathbf{O}^r\}, \cdots \{\mathbf{I}^{\rm PT}, \mathbf{O}^{\rm PT}\}$, we want to train fuzzy ARTMAP to map every input pattern of the training list to its corresponding output pattern. In order to achieve the aforementioned goal, we present the training list repeatedly to the Fuzzy ARTMAP architecture. That is we present $\mathbf{I}^1$ to $ART_a$ and $\mathbf{O}^1$ to $ART_b$, then $\mathbf{I}^2$ to $ART_a$ and $\mathbf{O}^2$ to $ART_b$,$\cdots$, and finally $\mathbf{I}^{\rm PT}$ to $ART_a$ and $\mathbf{O}^{\rm PT}$ to $ART_b$. This corresponds to *one list presentation*. The training list is presented as many times as it is necessary for fuzzy ARTMAP to correctly classify all the input patterns. The classification task is considered accomplished (i.e., learning is complete) when the weights do not change during a list presentation. The *performance phase* occurs when the trained fuzzy ARTMAP network is used to classify a list of test input patterns.

Two of the most important network parameters of the fuzzy ARTMAP architecture are the choice parameter and the vigilance parameter in the $ART_a$ module. For all of the experiments in this paper the values of these parameters are chosen to be zero (actually the choice parameter is a very small positive constant), because our objective is to focus on algorithms that do not require tuning of parameters. As mentioned previously, this choice of the network parameters leads to fuzzy ARTMAP network architectures of minimum size. The functionality of fuzzy ARTMAP is better illustrated by referring to the geometrical interpretation of the weights

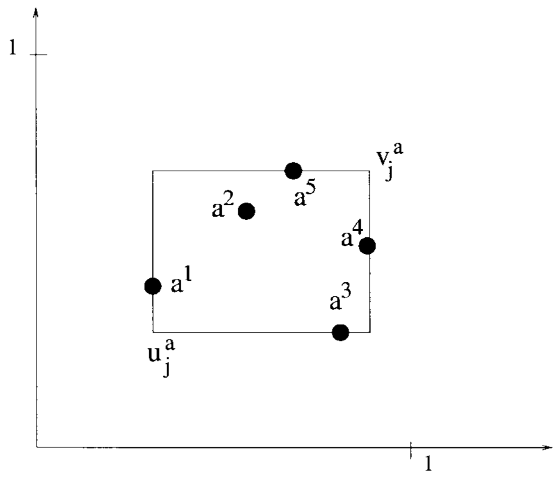


Fig. 2. The hyperrectangle which has coded the patterns $\mathbf{I}^1 = (\mathbf{a}^1, (\mathbf{a}^1)^c)$, $\mathbf{I}^2 = (\mathbf{a}^2, (\mathbf{a}^2)^c)$, $\mathbf{I}^3 = (\mathbf{a}^3, (\mathbf{a}^3)^c)$, $\mathbf{I}^4 = (\mathbf{a}^4, (\mathbf{a}^4)^c)$, and $\mathbf{I}^5 = (\mathbf{a}^5, (\mathbf{a}^5)^c)$.

in $ART_a$. As initially discussed in [6] and further elaborated in [7], every weight vector in $ART_a$ defines a hyperrectangle (hyperbox) in the input pattern space that includes all patterns that chose this weight vector as their representative during the training process. In Fig. 2, we show the hyperrectangle that the weight vector $\mathbf{w}_j^a$ (i.e., the weight vector corresponding to node $j$ in $F_2^a$) defines. Note that patterns $\mathbf{I}^1 = (\mathbf{a}^1, (\mathbf{a}^1)^c)$, $\mathbf{I}^2 = (\mathbf{a}^2, (\mathbf{a}^2)^c)$, $\mathbf{I}^3 = (\mathbf{a}^3, (\mathbf{a}^3)^c)$, $\mathbf{I}^4 = (\mathbf{a}^4, (\mathbf{a}^4)^c)$, and $\mathbf{I}^5 = (\mathbf{a}^5, (\mathbf{a}^5)^c)$ were coded by $\mathbf{w}_j^a = (\mathbf{u}_j^a, (\mathbf{v}_j^a)^c)$, where $\mathbf{u}_j^a$ and $\mathbf{v}_j^a$ correspond to the endpoints of the hyperrectangle that $\mathbf{w}_j^a$ defines. In Fig. 2, hyperrectangles are actually rectangles since the input patterns $\mathbf{I}$ are four-dimensional, and their components (the $\mathbf{a}$'s) are two-dimensional. After the training of fuzzy ARTMAP is completed, the weight vectors of committed nodes in $ART_a$ represent clusters of input patterns (hyperboxes) that are mapped to the same output pattern.

## III. The Ordering Algorithm

The purpose of the ordering algorithm of ordered fuzzy ARTMAP is to identify the order in which patterns should be presented during the training phase of fuzzy ARTMAP. This task is accomplished by following a systematic procedure that consists of three stages. Before we discuss these stages let us first define the parameters $n_{\text{clust}}$, *PT*, and the set $S_T$ that appear in the algorithm's description. In this paper, the parameter $n_{\text{clust}}$ is taken to be either equal to the number of distinct classes or equal to one more than the number of distinct classes associated with the pattern classification task. The parameter *PT* stands for the number of input–output pairs in the training list. Finally, $S_T$ is, prior to the application of the ordering algorithm, the set of all training input patterns. In Stage 1, we choose the first pattern to be presented. This pattern corresponds to the first cluster center of the training input patterns. In Stage 2, we choose the next $(n_{\text{clust}} - 1)$ patterns to be presented. These patterns correspond to the next $(n_{\text{clust}} - 1)$ cluster centers of the training input patterns, and are identified using the *Max–Min* clustering algorithm [5]. In Stage 3, we choose the remaining $(PT - n_{\text{clust}})$ patterns to be presented. These patterns are chosen according to the minimum Euclidean distance criterion from the $n_{\text{clust}}$ cluster centers defined in Stages 1 and 2. Below, we describe in more detail each of these stages.

*Stage 1—The First Pattern:* For each pattern $\mathbf{I} = (a_1, \cdots a_{M_a}, a_{M_a+1}, \cdots, a_{2M_a})$ in the training set we compute

$$\sum_{i=1}^{M_a} \left| a_{M_a+i} - a_i \right|. \tag{3}$$

The pattern from the training set that maximizes the above sum is the first pattern presented to ordered fuzzy ARTMAP, and the first cluster center used in Stage 2. The training pattern that maximizes the above sum is removed from the training set $S_T$. To understand how the first pattern is produced in the first stage of the ordering algorithm we present a simple example in Appendix I. The following two stages of the ordering procedure involve calculation of Euclidean distances among patterns in the training set. In the calculation of these distances only the first $M_a$ components of the input patterns are used (i.e., the $\mathbf{a}$ portion of the $\mathbf{I}$ vector). To avoid switching back and forth between the $\mathbf{a}$ and $\mathbf{I}$ notation, we refer to these distances as the distances among the $\mathbf{I}$'s.

*Stage 2—The Next $(n_{\text{clust}} - 1)$ Patterns:* This stage uses the Max–Min clustering algorithm to define $(n_{\text{clust}} - 1)$ appropriate cluster centers (patterns), which constitute the next $(n_{\text{clust}} - 1)$ input patterns to be presented during the training phase of ordered fuzzy ARTMAP. The steps followed to define these cluster centers are as follows. The index $r$, initialized and updated in the step-by-step description of Stage 2, corresponds to the number of clusters that have been identified, at various points, during the implementation of Stage 2.

1) Denote the first cluster center (input pattern) identified in Stage 1 by $\mathbf{I}_O^1$, and initialize the index $r$ to one.
2) Compute the Euclidean distance of every input pattern in the training set $S_T$ to the $k$th cluster center, and find the minimum one, $d_{\min}^I$. That is,

$$d_{\min}^I = \min_{1 \le k \le r} \left\{ dist\left(\mathbf{I}, \mathbf{I}_O^k\right) \right\}. \tag{4}$$

3) Find the input pattern from the training set $S_T$ that maximizes $d_{\min}^I$, $I \in s_T$. Designate this input pattern by the generic name $\mathbf{I}$. The next cluster center, designated by $\mathbf{I}_O^{r+1}$, is equal to $\mathbf{I}$, that is $\mathbf{I}_O^{r+1} = \mathbf{I}$. This cluster center constitutes the next input pattern to be presented during the training phase of ordered fuzzy ARTMAP. Increment $r$ by one, and eliminate input pattern $\mathbf{I}$ from the training set $S_T$.
4) If $r = n_{\text{clust}}$ this stage is completed; otherwise, go to Step 2).

At the end of Stages 1 and 2, we have identified $n_{\text{clust}}$ cluster centers that correspond to the input patterns $\mathbf{I}_O^r$, $1 \le r \le n_{\text{clust}}$, of the training set. The next stage identifies the order according to which the remaining input patterns should be presented to the ordered fuzzy ARTMAP.

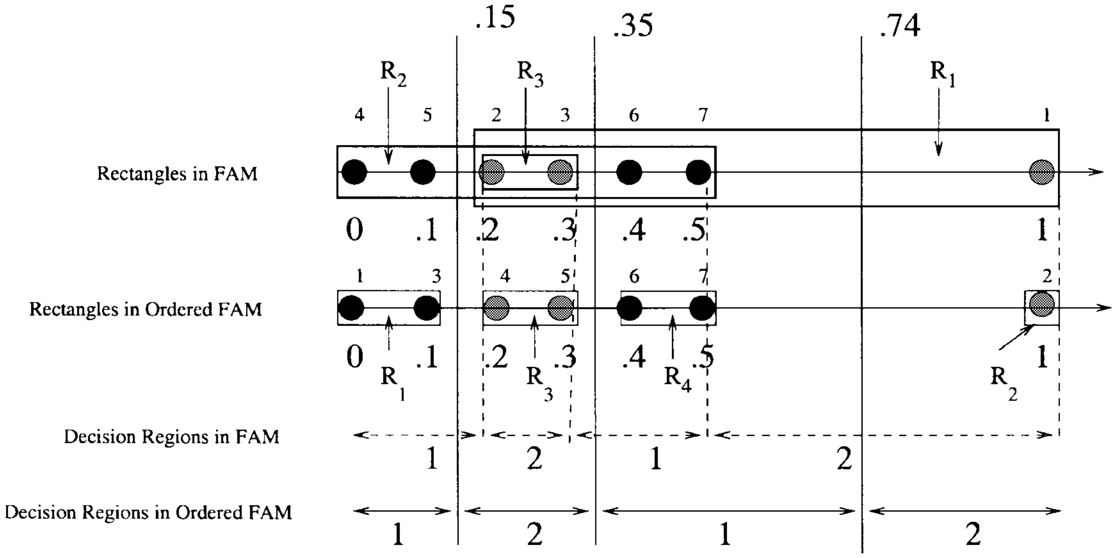


Fig. 3. Rectangles and decision regions of fuzzy ARTMAP (FAM) and ordered fuzzy ARTMAP (ordered FAM) for Example 1.

*Stage 3—The Remaining* $(PT - n_{\text{clust}})$ *Input Patterns:* The steps followed in this stage are as follows.

1) Set index $r$ to $n_{\text{clust}}$. The patterns in the training set $S_T$ are all of the training input patterns except the ones identified as cluster centers in Stages 1 and 2.
2) Calculate the Euclidean distance of every pattern $\mathbf{I}$ in the set $S_T$ to the $n_{\text{clust}}$ cluster centers.
3) Find the minimum of these distances. Assume that it corresponds to input pattern $\mathbf{I}$. This pattern is the next in sequence input pattern to be presented in the training phase of fuzzy ARTMAP. Eliminate $\mathbf{I}$ from the set $S_T$, set $\mathbf{I}_O^{r+1} = \mathbf{I}$, and increment $r$.
4) If $r = PT$ this stage is complete; otherwise, go to Step 2).

After the end of Stage 3, we have identified the ordered set of patterns $\mathbf{I}_O^1, \mathbf{I}_O^2, \cdots, \mathbf{I}_O^{PT}$. This is the order according to which the patterns in the training set will be presented to the ordered fuzzy ARTMAP. The corresponding outputs of this ordered sequence of input patterns are the outputs from the training list that these input patterns need to be mapped to. For example, if $\mathbf{I}_O^1 = \mathbf{I}^2$, then $\mathbf{I}_O^1$'s corresponding output is $\mathbf{O}^2$. It is worth mentioning that the ordering that the ordering algorithm produces is independent of any permutations of the input training patterns. Proof of this statement is provided in Appendix II (Theorem 1).

## IV. Examples—Motivation

In order to better understand the differences between a random order of training pattern presentation and the proposed fixed order of training pattern presentation, we present some illustrative examples. Once the examples are presented it will be easier to explain the motivation for our work. In Example 1 (see Fig. 3) the data $\{0, 0.1, 0.4, 0.5\}$ belong to Class 1, while the data $\{0.2, 0.3, 1\}$ belong to Class 2. The ordering algorithm, with $n_{\text{clust}} = 2$, computed the following order for training pattern presentation: $\{0, 1.0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. Note that Stage 1 in this example identified pattern 0 as the first pattern to be presented (the closest point to one of the corners of the input pattern space), Stage 2 identified pattern 1 as the second pattern to be presented, and Stage 3 identified the order for the rest of the patterns in the training set. The training patterns in fuzzy ARTMAP were presented in the following order: $\{1.0, 0.2, 0.3, 0, 0.1, 0.4, 0.5\}$. Observe that in Fig. 3, the numbers above the black and gray circles indicate the order of their presentation in the training phases of fuzzy ARTMAP and ordered fuzzy ARTMAP. After training is over in fuzzy ARTMAP, rectangles $R_1$ (with endpoints 0.2 and one), $R_2$ (with endpoints zero and 0.5), and $R_3$ (with endpoints 0.2 and 0.3) have been created. After training is over in ordered fuzzy ARTMAP, rectangles $R_1$ (with endpoints zero and 0.1), $R_2$ (containing the datum 1.0), $R_3$ (with endpoints 0.2 and 0.3), and $R_4$ (with endpoints 0.4 and 0.5) have been created. The decision regions for fuzzy ARTMAP (shown in Fig. 3), classify test data between 0.2 and 0.3 and between 0.5 and one, as Class 2, and the rest of the test data as Class 1. On the other hand, the decision regions for the ordered fuzzy ARTMAP (shown in Fig. 3) classify the test data between zero and 0.15 and between 0.35 and 0.74 as Class 1, and the rest of the data as Class 2. In this example, ordered fuzzy ARTMAP has reduced the effect of the nonrepresentative rectangle $R_1$. This reduction was done by shrinking the decision region $[0.5, 1]$ to the region $[0.74, 1]$. Rectangle $R_1$ of fuzzy ARTMAP is nonrepresentative of the data because it forces the network to classify all test data in the interval $[0.5, 1]$ as data belonging to Class 2, despite the fact that there is no Class 2 training data within the interval $[0.3, 1]$.

In Example 2 (see Fig. 4), the data $\{0, 0.05, 0.1, 0.4, 0.45, 0.51\}$ belong to Class 1, while the data $\{0.2, 0.25, 0.3, 0.6, 0.65, 0.7\}$ belong to Class 2. The ordering algorithm, with $n_{\text{clust}} = 2$, identified the following order of training pattern presentation: $\{0, 0.7, 0.05, 0.65, 0.1, 0.6, 0.51, 0.2, 0.25, 0.45, 0.3, 0.4\}$ for ordered fuzzy ARTMAP. Note that Stage 1 in this example identified pattern 0 as the first pattern to be presented (the closest point to one of the corners

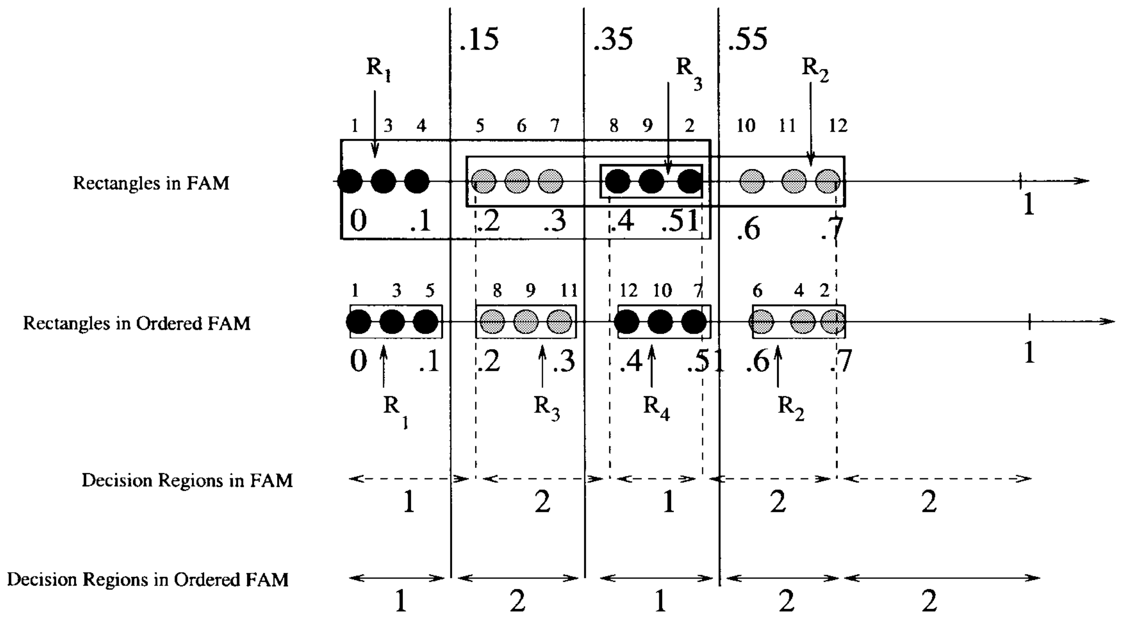


Fig. 4. Rectangles and decision regions of fuzzy ARTMAP (FAM) and ordered fuzzy ARTMAP (ordered FAM) for Example 2.

of the input pattern space), Stage 2 identified pattern 0.7 as the second pattern to be presented, and Stage 3 identified the order for the rest of the patterns in the training set. The training patterns in fuzzy ARTMAP were presented in the following order: $\{0, 0.51, 0.15, 0.1, 0.2, 0.25, 0.3, 0.4, 0.45, 0.6, 0.65, 0.7\}$. Observe that in Fig. 4, the numbers above the black and gray circles indicate the order of their presentation during the training phases of fuzzy ARTMAP and ordered fuzzy ARTMAP. After training is over in fuzzy ARTMAP, rectangles $R_1$ (with endpoints zero and 0.51), $R_2$ (with endpoints 0.2 and 0.7), and $R_3$ (with endpoints 0.4 and 0.51) have been created. After training is over in ordered fuzzy ARTMAP, rectangles $R_1$ (with endpoints zero and 0.1), $R_2$ (with endpoints 0.6 and 0.7), $R_3$ (with endpoints 0.2 and 0.3), and $R_4$ (with endpoints 0.4 and 0.51) have been created. The decision regions for fuzzy ARTMAP (shown in Fig. 4) classify test data between zero and 0.2 and between 0.4 and 0.51 as Class 1, and the rest of the data as Class 2. On the other hand, the decision regions for the ordered fuzzy ARTMAP (shown in Fig. 4) classify the test data between zero and 0.15 and the test data between 0.35 and 0.55 as Class 1, and the rest of the data as Class 2. In this example, ordered fuzzy ARTMAP has reduced the overlap among rectangles that lead to different outputs [see desired Property 8) of a pattern classifier].

The major motivation for our work was the design of a fuzzy ARTMAP algorithm that is independent of the tuning of parameters, and achieves good generalization by avoiding excessive experimentation. The dependence of fuzzy ARTMAP on the choice parameter and the vigilance parameter is an inherent characteristic of the algorithm. Choosing these parameters equal to zero frees the experimenter from the tedious task of optimizing the network performance with respect to these two parameters. With the choice parameter and the vigilance parameter chosen equal to zero, one ends up with a fuzzy ARTMAP algorithm that exhibits a significant variation in generalization performance for different orders of training pattern presentations. Furthermore, it is not an easy task to guess which one of the exceedingly large number of orders of pattern presentations exhibits the best generalization. Our assumption was that orders of pattern presentation that create unnecessarily large rectangles (e.g., rectangle $R_1$ of Example 1 for fuzzy ARTMAP), that force assumptions about the data where training data do not exist (e.g., region $[0.5, 1]$ for fuzzy ARTMAP in Example 1), give credibility to possible outlier data (e.g., datum "1" in Example 1 for fuzzy ARTMAP), and include in their regions data that belong to more than one class (e.g., Examples 1 and 2), would not lead to good generalization performance. So, the idea was to address this problem by spreading out enough initial clusters at locations where data exist (e.g., clusters 1 and 2 in Examples 1 and 2), and then present the rest of the training data in order of closest distance to these initial clusters. The examples presented above justified the validity of our approach.

## V. Experimental Results—Comparisons

In the following sections, we describe the databases used to compare fuzzy ARTMAP and ordered fuzzy ARTMAP, define performance measures, and compare the two algorithms by conducting appropriate experiments.

### *A. Databases*

In order to demonstrate the superior performance of ordered fuzzy ARTMAP as compared to fuzzy ARTMAP, we chose to conduct experiments on a number of databases extracted from the UCI repository database [8]. The databases chosen from the repository were: Iris [9], Wine [10], Sonar [11], Diabetes [12], Breast [13], Balance [14], Bupa [8], Cars [8], and Glass [8]. The Sonar, Diabetes, Breast, and Bupa are two-class classification problems, the Iris, Wine, Balance are three-class

TABLE I
GENERALIZATION PERFORMANCES OF THE FUZZY ARTMAP AND THE ORDERED FUZZY ARTMAP WITH $n_{\text{clust}}$ = (NUMBER OF CLASSES) + 1

| | Fuzzy ARTMAP | | | | Ordered Fuzzy ARTMAP | |
|---|---|---|---|---|---|---|
| Database | Worst Gen. | Best Gen. | avg.Gen. | std.dev. | $n_{clust}$ | Gen. |
| Sonar | 63.77 | 78.26 | 70.58 | 4.15 | 3 | 79.96 |
| Diabetes | 61.57 | 70.98 | 66.63 | 2.57 | 3 | 69.80 |
| Breast | 93.10 | 96.12 | 94.35 | 0.95 | 3 | 94.39 |
| Bupa | 47.37 | 63.16 | 56.84 | 4.22 | 3 | 57.01 |
| Iris | 89.58 | 95.83 | 95.00 | 1.91 | 4 | 97.92 |
| Wine | 91.38 | 98.28 | 95.69 | 2.70 | 4 | 98.27 |
| Balance | 71.63 | 78.85 | 75.91 | 2.42 | 4 | 75.48 |
| Cars | 63.21 | 69.29 | 66.43 | 2.13 | 5 | 67.50 |
| Glass | 57.97 | 76.81 | 63.77 | 6.18 | 7 | 69.56 |

classification problems, the Cars is a four-class classification problem, and finally the Glass is a six-class classification problem. Each of these datasets was split randomly into a training set (2/3 of the data) and a test set (1/3 of the data). The percentage of data from each class in the training and test set reflected the percentage of data from each class in the entire dataset. The number of the data used in the training set for Iris, Wine, Sonar, Diabetes, Breast, Balance, Bupa, Cars and Glass were 102, 120, 139, 513, 467, 417, 231, 566, 145, respectively. The number of the data used in the test set for Iris, Wine, Sonar, Diabetes, Breast, Balance, Bupa, Cars and Glass were 48, 58, 69, 255, 232, 208, 114, 280, 69, respectively. The dimensionality of the input patterns for Iris, Wine, Sonar, Diabetes, Breast, Balance, Bupa, Cars and Glass were 4, 13, 60, 8, 9, 4, 6, 18, 9, respectively. More detailed descriptions of each one of these databases can be found in the references.

## *B. Measures of Performance*

One of the performance measures used to compare ordered fuzzy ARTMAP and fuzzy ARTMAP is *generalization*. The generalization performance of a network is defined as the percentage of patterns in the test set that are correctly classified by a trained network. Since the performance of fuzzy ARTMAP depends on the order of pattern presentation in the training set, ten different random orders of pattern presentation were investigated, and performance measures such as the average generalization performance, the worst generalization performance, the best generalization performance, and the standard deviation of the generalization performance were produced for fuzzy ARTMAP. Other measures of comparison of ordered fuzzy ARTMAP and fuzzy ARTMAP are the sizes of the networks that these two algorithms create, and the numbers of operations required by ordered fuzzy ARTMAP as compared to the number of operations required by fuzzy ARTMAP.

## *C. Comparisons of Ordered Fuzzy ARTMAP and Fuzzy ARTMAP*

The only weak link in the procedure that finds an ordered sequence of training patterns for ordered fuzzy ARTMAP is that the number of clusters parameter ($n_{\text{clust}}$) must be specified in Stage 2. Our experimental results have shown that a good rule of thumb for choosing the number of clusters $n_{\text{clust}}$, is the *number of classes* or one more than the *number of classes* in the data set. This rule of thumb tends to produce an ordered fuzzy ARTMAP with the best generalization performance.

TABLE II
AVERAGE NETWORK SIZE FOR FUZZY ARTMAP AND NETWORK SIZE OF ORDERED FUZZY ARTMAP WITH $n_{\text{clust}}$ = (NUMBER OF CLASSES) + 1

| | Fuzzy ARTMAP | Ordered Fuzzy ARTMAP | |
|---|---|---|---|
| Database | Avg. Net Size | $n_{\text{clust}}$ | Net Size |
| Sonar | 6 | 3 | 5 |
| Diabetes | 43 | 3 | 44 |
| Breast | 8 | 3 | 9 |
| Bupa | 31 | 3 | 31 |
| Iris | 5 | 4 | 4 |
| Wine | 4 | 4 | 6 |
| Balance | 79 | 4 | 120 |
| Cars | 46 | 5 | 56 |
| Glass | 27 | 7 | 30 |

In Table I, we show generalization performance comparisons between fuzzy ARTMAP and ordered fuzzy ARTMAP when $n_{\text{clust}}$ is chosen to be equal to one more than the *number of classes*. Looking at the results we observe the following: The generalization performance of the ordered fuzzy ARTMAP is *better than the worst* generalization performance of fuzzy ARTMAP by 16.19%, 11.59%, 9.64%, 8.34%, 8.23%, 6.89%, 4.29%, 3.85%, and 1.29% for the Sonar, Glass, Bupa, Iris, Diabetes, Wine, Cars, Balance, and Breast databases, respectively. The generalization performance of the ordered fuzzy ARTMAP is *better than the average* generalization performance of fuzzy ARTMAP by 9.38%, 5.79%, 3.17%, 2.92%, 2.58%, 1.07%, 0.17%, 0.04%, and −0.43% for the Sonar, Glass, Diabetes, Iris, Wine, Cars, Bupa, Breast, and Balance databases, respectively. The generalization performance of the ordered fuzzy ARTMAP is *better than the best* generalization performance of fuzzy ARTMAP by 2.09%, 1.7%, −0.01%, −1.18%, −1.73%, −1.79%, −3.37%, −6.15%, and −7.25% for the Iris, Sonar, Wine, Diabetes, Breast, Cars, Balance, Bupa, and Glass databases, respectively. Negative percentages imply that the corresponding fuzzy ARTMAP generalization performance (worst, average, or best) is better than the ordered fuzzy ARTMAP generalization performance.

In Table II, we show the size of the network that ordered fuzzy ARTMAP created and the average size of the network that fuzzy ARTMAP created. It is worth pointing out that the size of the neural-network architectures that ordered fuzzy ARTMAP creates range between 0.80 and 1.52 of the average size of the network architectures that fuzzy ARTMAP creates. Also, for most of the databases (Sonar, Diabetes, Breast, Bupa, Iris and Glass) the ordered fuzzy ARTMAP network size is either smaller or approximately equal to the average fuzzy ARTMAP network size. In Appendix III we perform a very detailed analysis of the number of operations required by the ordering algorithm and the average number of operations required by the training phase of fuzzy ARTMAP. This analysis shows that in both cases the number of operations required is $O(PT)$, where the constant of proportionality in the ordering algorithm is approximately equal to $n_{\text{clust}}^2$, while the constant of proportionality in fuzzy ARTMAP is approximately equal to $\sum_{e=1}^{E} n_e$, where $n_e$ is the average number of categories in $F_2^a$ during the $e$th epoch of training, and $E$ is the average number of epochs needed by fuzzy ARTMAP to learn the required task. As can be seen in Table II, there are databases (e.g., Diabetes, Bupa, Balance, Cars), where the constant $n_{\text{clust}}^2$ could be a small fraction of $\sum_{e=1}^{E} n_e$, and as a result the operations required by ordered fuzzy ARTMAP could be a small fraction of the operations required by the training phase of fuzzy ARTMAP. It is worth pointing out that we have repeated the experiments with the above databases for three different collections of training/test sets beyond the ones for which results were reported in Tables I and II above. The numerical results obtained with these new training/test data sets were slightly different than the ones shown in Tables I and II, but the conclusions obtained from these results were of the same nature as the conclusions derived from Tables I and II.

Since ordered fuzzy ARTMAP does not have the on-line capability of fuzzy ARTMAP, it is fair to compare the performance of ordered fuzzy ARTMAP with other classification techniques that have appeared in the literature. In order to make these comparisons we rely on the experimental results produced by Joshi *et al.* [15]. These authors compare a number of neuro-fuzzy, machine learning, and statistical techniques on a variety of databases. The measure of comparison is the generalization performance. The classical machine learning algorithms used in this comparison were ID3, HOODG, Const, IB, C4.5, Bayes, oneR, Aha-IB, Dis-Bayes, and OC1-Inducer. The statistical techniques used were regression models and discriminant analysis. The neuro-fuzzy techniques utilized consisted of backpropagation, backpropagation with momentum, Quickprop, R-Prop, LVQ1, OLVQ1, LVQ2, and LVQ3. The databases used were Iris, Pythia, Soybean, Glass, Ionosphere, ECG and Wine. The data were split (as is the the case in this paper) into a training set and a test set (2/3 and 1/3 of the whole dataset, respectively). The reported results correspond to the best set of parameters for each one of the aforementioned techniques [15]. For the Iris database, the range of the generalization performances achieved by the above techniques was from a minimum of 78.5% to a maximum of 95.7%. Ordered fuzzy ARTMAP achieved a generalization performance of 97.92% without any parameter optimization. For the Wine database, the range of the generalization performances reported by using the above techniques was from a minimum of 90.2% to a maximum of 100%. The 100% performance was achieved only by one algorithm (Simpson's); the rest of the techniques had a best performance of 98%. Ordered fuzzy ARTMAP achieved a generalization performance of 98.27% without any parameter optimization. For the Glass database the reported generalization performance ranged from a minimum of 83.7% to a maximum of 95.13%. The corresponding ordered fuzzy ARTMAP generalization performance was 69.56%. Note that the range of fuzzy ARTMAP generalization performances for the Glass database was from a minimum of 57.97% to a maximum of 76.81%. Hence, fuzzy ARTMAP does not perform very well with this database and the poor generalization reported for this database should not necessarily be attributed to any limitations of the proposed ordering algorithm. Actually, from [15], we see the best generalization performances (95.13%) attained for the Glass is attributed to Simpson's algorithm, which is an example of an ART-like architecture; although Simpson's algorithm was allowed to experiment with parameters in order to optimize generalization performance. The objective in this paper was to propose an ordering algorithm that leads to a fuzzy ARTMAP network with good generalization performance (at least for most databases) without having to resort to excessive experimentation to tune the network's parameters. We believe that our experimental results, and the above comparisons with other techniques support the validity of our approach.

## VI. Review—Conclusions

In this paper we introduced a procedure, referred to as the ordering algorithm, that identifies a fixed order of training pattern presentation for fuzzy ARTMAP. The ordering algorithm is based on the Max–Min clustering algorithm. The combination of the ordering algorithm and fuzzy ARTMAP is called ordered fuzzy ARTMAP. Experiments with nine different classification problems have shown that ordered fuzzy ARTMAP attains a superior generalization performance as compared to the average performance of fuzzy ARTMAP, and in certain cases as good as, or better than the best fuzzy ARTMAP generalization performance. The average and best generalization performances are obtained over a fixed number of experiments with fuzzy ARTMAP corresponding to different orders of training pattern presentations. We also demonstrated that under mild conditions on the pattern classification tasks, the operations required by the ordering algorithm is a fraction of the operations required by the training phase of fuzzy ARTMAP for a single order of training pattern presentation. Furthermore, the sizes of the network architectures that ordered fuzzy ARTMAP creates are comparable to the average size of the network architectures that fuzzy ARTMAP creates. Finally, the proposed procedure for ordering the training data may also be applied to other ART-type architectures, such as Simpson's Min–Max architecture [1], the LAPART architecture of Healy *et al.* [16], and the ARTEMAPQ and ARTMAP-IC architectures of Carpenter's *et al.* [17].

## Appendix I

In order to explain the choice of the first pattern in the ordering algorithm, let us consider Example 2 of Fig. 4. Example 2 was fully described in Section IV. In this example, the data $\{0, 0.05, 0.1, 0.4, 0.45, 0.51\}$ belong to Class 1, while the data $\{0.2, 0.25, 0.3, 0.6, 0.65, 0.7\}$ belong to Class 2. If we apply (3) to these datapoints: Datum "0" with representation $(a_1, a_2) = (0, 1)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |1 - 0| = 1$, Datum "0.05" with representation $(a_1, a_2) = (0.05, 0.95)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.95 - 0.05| = 0.9$, Datum "0.1" with representation $(a_1, a_2) = (0.1, 0.9)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.9 - 0.1| = 0.8$, Datum "0.2" with representation $(a_1, a_2) = (0.2, 0.8)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.8 - 0.2| = 0.6$, Datum "0.25" of with representation $(a_1, a_2) = (0.25, 0.75)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.75 - 0.25| = 0.5$, Datum "0.3" with representation $(a_1, a_2) = (0.3, 0.7)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.7 - 0.3| = 0.4$, Datum "0.4" with representation $(a_1, a_2) = (0.4, 0.6)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.6 - 0.4| = 0.2$, Datum "0.45" with representation $(a_1, a_2) = (0.45, 0.55)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.55 - 0.45| = 0.1$, Datum "0.51" of with representation $(a_1, a_2) = (0.51, 0.49)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.49 - 0.51| = 0.02$, Datum "0.6" with representation $(a_1, a_2) = (0.6, 0.4)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.4 - 0.6| = 0.2$, Datum "0.65" with representation $(a_1, a_2) = (0.65, 0.35)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.35 - 0.65| = 0.3$, Datum "0.7" with representation $(a_1, a_2) = (0.7, 0.3)$ yields $\sum_{i=1}^{1} |a_{i+1} - a_i| = |0.3 - 0.7| = 0.4$. From the above calculations it is easy to see that the datum that maximizes $\sum_{i=1}^{M_a} |a_{i+M_a} - a_i|$ is datum "0," one of the endpoints of the interval $[0, 1]$. In higher dimensions, we expect Stage 1 to give us a datapoint that is close to one of the vertices of the hypercube in which the input patterns reside.

## APPENDIX II

*Theorem 1:* The order sequence of the training input patterns that the ordering algorithm produces is independent of any permutations of their indexes.

*Proof:* We can prove Theorem 1 in the general case, when dealing with an arbitrary number of training input patterns and for an arbitrary choice of $n_{\text{clust}}$, but have not provided this proof due to the cumbersome notation involved. We believe that presenting an example which verifies the theorem is more enlightening. Let us, therefore, refer to Example 1 of Fig. 3 (see Section IV in the main text). We will consider two distinct permutations of the training input patterns. Permutation 1 assigns indexes in increasing order, first to input patterns of Class 1, and then to input patterns of Class 2. Within a class indexes are assigned in increasing order, according to increasing distance from the vertex "0." Hence, for Permutation 1 we designate the collection of input patterns as follows:

$$\begin{aligned} \mathbf{I}^1 &= (0.0, 1.0) \\ \mathbf{I}^2 &= (0.1, 0.9) \\ \mathbf{I}^3 &= (0.4, 0.6) \\ \mathbf{I}^4 &= (0.5, 0.5) \\ \mathbf{I}^5 &= (0.2, 0.8) \\ \mathbf{I}^6 &= (0.3, 0.7) \\ \mathbf{I}^7 &= (1.0, 0.0). \end{aligned} \tag{5}$$

Permutation 2 assigns indexes in increasing order first to input patterns of Class 2, and then to input patterns, of Class 1. Within a class indexes are assigned in increasing order, according to increasing distance from the vertex "0." Hence, for Permutation 2 we designate the collection of input patterns as follows:

$$\begin{aligned} \hat{\mathbf{I}}^1 &= (0.2, 0.8) \\ \hat{\mathbf{I}}^2 &= (0.3, 0.7) \\ \hat{\mathbf{I}}^3 &= (1.0, 0.0) \\ \hat{\mathbf{I}}^4 &= (0.0, 1.0) \\ \hat{\mathbf{I}}^5 &= (0.1, 0.9) \\ \hat{\mathbf{I}}^6 &= (0.4, 0.6) \\ \hat{\mathbf{I}}^7 &= (0.5, 0.5). \end{aligned} \tag{6}$$

We denote by $S_T$ the set of training input patterns of Permutation 1 (initially taken to be all the $\mathbf{I}$'s), and by $\hat{S}_T$ the set of training input patterns of Permutation 2 (initially taken to be all the $\hat{\mathbf{I}}$'s). The application of (3) of Stage 1 of the ordering algorithm to the input patterns, designated in Permutation 1, produces the following numbers in order of increasing index: 1.0, 0.8, 0.2, 0.0, 0.6, 0.4, 1.0. The maximum number (1.0) corresponds to patterns $\mathbf{I}^1 = (0.0, 1.0)$ and $\mathbf{I}^7 = (1.0, 0.0)$; we choose (arbitrarily) pattern $\mathbf{I}^1 = (0.0, 1.0)$ since it is the input pattern closest to vertex "0." Pattern $\mathbf{I}^1 = (0.0, 1.0)$ is eliminated from the training set $S_T$. The application of (3) of Stage 1 of the ordering algorithm to the input patterns, designated in Permutation 2, produces the following numbers in order of increasing index: 0.6, 0.4, 1.0, 1.0, 0.8, 0.2, 0.0. The maximum number (1.0) corresponds to patterns $\hat{\mathbf{I}}^3 = (1.0, 0.0)$ and $\hat{\mathbf{I}}^4 = (0.0, 1.0)$; we choose (arbitrarily) pattern $\hat{\mathbf{I}}^4 = (0.0, 1.0)$ since it is the input pattern closest to vertex "0." Pattern $\hat{\mathbf{I}}^4 = (0.0, 1.0)$ is eliminated from the training set $\hat{S}_T$. Thus, for both permutations, the application of Stage 1 of the ordering algorithm led us to choose the same input pattern to be presented first to ordered fuzzy ARTMAP.

The application of (4) of Stage 2 of the ordering algorithm to the input patterns, designated in Permutation 1, produces the following distances in order of increasing index (remember that the pattern with index 1 in Permutation 1 was eliminated from $S_T$ in Stage 1): 0.1, 0.4, 0.5, 0.2, 0.3, 1.0. The maximum number (1.0) corresponds to pattern $\mathbf{I}^7 = (1.0, 0.0)$, and as result this is the pattern chosen to be presented next to ordered fuzzy ARTMAP. Pattern $\mathbf{I}^7 = (1.0, 0.0)$ is eliminated from the training set $S_T$. The application of (4) of Stage 2 of the ordering algorithm to the input patterns, designated in Permutation 2, produces the following distances in order of increasing index (remember that the pattern with index 4 in Permutation 2 was eliminated from $\hat{S}_T$ in Stage 1): 0.2, 0.3, 1.0, 0.1, 0.4, 0.5. The maximum number (1.0) corresponds to pattern $\hat{\mathbf{I}}^3 = (1.0, 0.0)$, and as result this is the pattern chosen to be presented next to ordered fuzzy ARTMAP. Pattern $\hat{\mathbf{I}}^3 = (1.0, 0.0)$ is eliminated from the training set $\hat{S}_T$. Since $n_{\text{clust}}$ for this example equals 2, Stage 2 is completed at this point.

The application of Stage 3 of the ordering algorithm to the input patterns designated in Permutation 1 produces distances 0.1, 0.4, 0.5, 0.2, 0.3 from the first cluster point, and distances 0.9, 0.6, 0.5, 0.8, 0.7 from the second cluster point; distances are reported in order of increasing index for the Permutation 1 patterns. Hence, according to the Stage 3 rules, the remaining input patterns will be presented in the order $\mathbf{I}^2 = (0.1, 0.9)$, $\mathbf{I}^5 = (0.2, 0.8)$, $\mathbf{I}^6 = (0.3, 0.7)$, $\mathbf{I}^3 = (0.4, 0.6)$, and $\mathbf{I}^4 = (0.5, 0.5)$. The application of Stage 3 of the ordering algorithm to the input patterns designated in Permutation 2 produces distances 0.2, 0.3, 0.1, 0.4, 0.5 from the first cluster point, and distances 0.8, 0.7, 0.9, 0.6, 0.5 from the second cluster point, where distances are reported in order of increasing index for the Permutation 2 patterns. According to the Stage 3 rules, the remaining input patterns will be presented in the order $\hat{\mathbf{I}}^5 = (0.1, 0.9)$, $\hat{\mathbf{I}}^1 = (0.2, 0.8)$, $\hat{\mathbf{I}}^2 = (0.3, 0.7)$, $\hat{\mathbf{I}}^6 = (0.4, 0.6)$, and $\hat{\mathbf{I}}^7 = (0.5, 0.5)$. Consequently, both Permutations 1 and 2 lead to the same ordering of the training input patterns, a result which verifies the validity of Theorem 1. □

## Appendix III

The major operations involved in the computations associated with the ordering algorithm are: addition, subtraction, multiplication, absolute value, and comparison. We assume these operations are equally expensive. This is in fact a reasonable assumption for CISC (complex instruction set) computers. The operations required by the ordering algorithm will be separated into two categories: Operations that are required to calculate the necessary distances, and operations required to find the minimum or maximum of these distances.

### *Operations for the Ordering Algorithm Required to Calculate Distances*

*Stage 1:* $3M_a PT$ operations to calculate $\sum_{i=1}^{M_a} |a_{i+M_a} - a_i|$ for the *PT* training input patterns. Each pattern requires $M_a$ subtractions, $M_a$ absolute value operations and $M_a$ additions.

*Stage 2:* For the second cluster center we need $3M_a(PT - 1)$ calculations to compute the distances of $(PT - 1)$ training input patterns from the first cluster center. The per pattern operations are $3M_a$ because in order to compute the Euclidean distance of two patterns of dimensionality $M_a$ we need $M_a$ subtractions, $M_a$ multiplications and $M_a$ additions. For the third cluster center we need $3M_a 2(PT - 2)$ calculations to compute the distances of $(PT - 2)$ training input patterns from cluster centers 1 and 2. Eventually, for the $n_{\text{clust}}$ cluster centers we need $3M_a(n_{\text{clust}} - 1)(PT + 1 - n_{\text{clust}})$ operations to compute the distances of $(PT + 1 - n_{\text{clust}})$ training input patterns from $(n_{\text{clust}} - 1)$ cluster centers. Hence, overall we need

$$3M_a[(PT - 1) + 2(PT - 2) + \cdots + (n_{\text{clust}} - 1)(PT + 1 - n_{\text{clust}})] \tag{7}$$

operations to calculate the necessary distances in Stage 2 of the Ordering Algorithm.

*Stage 3:* To calculate the distances of $(PT - n_{\text{clust}})$ training input patterns from $n_{\text{clust}}$ cluster centers we need $3M_a n_{\text{clust}}(PT - n_{\text{clust}})$ operations. Combining the distance-related operations for Stages 1, 2, and 3 of the ordering algorithm we see that we need

$$3M_a[PT + (PT - 1) + 2(PT - 2) + \cdots + n_{\text{clust}}(PT - n_{\text{clust}})] \tag{8}$$

operations, or approximately (if $n_{\text{clust}} \ll PT$)

$$\begin{aligned} &3M_a PT[1 + 1 + 2 + \cdots + n_{\text{clust}}] \\ &\quad = 3M_a PT[1 + n_{\text{clust}}(1 + n_{\text{clust}})/2] \end{aligned} \tag{9}$$

operations.

### *Operations for the Ordering Algorithm Required to Calculate Max and/or Min of Distances*

*Stage 1:* No maximum or minimum distance calculations are required in this stage.

*Stage 2:* For the second cluster center we need $(PT - 2)$ operations to find the minimum of $(PT - 1)$ distances. For the third cluster center we need $2(PT - 3)$ operations (comparisons) to find the minimum of $(PT - 2)$ distances from cluster center 1 and the minimum of $(PT - 2)$ distances from cluster center 2. Then, we need one operation to find the maximum of these two minimum distances. Eventually for the $n_{\text{clust}}$th cluster center we need $(n_{\text{clust}} - 1)(PT - n_{\text{clust}})$ to find the minimum of $(PT + 1 - n_{\text{clust}})$ distances from cluster center 1, the minimum of $(PT + 1 - n_{\text{clust}})$ distances from cluster center 2, and eventually the minimum of $(PT + 1 - n_{\text{clust}})$ distances from cluster center $n_{\text{clust}} - 1$. Then, we need $(n_{\text{clust}} - 2)$ operations to find the maximum of these $(n_{\text{clust}} - 1)$ minimum distances. Hence, overall we need

$$\begin{aligned} &[(PT - 2) + 2(PT - 3) + \cdots + (n_{\text{clust}} - 1)(PT - n_{\text{clust}})] \\ &\quad + [0 + 1 + \cdots + (n_{\text{clust}} - 2)] \end{aligned} \tag{10}$$

operations, or approximately (if $n_{\text{clust}} \ll PT$)

$$PT[1 + 2 + \cdots + (n_{\text{clust}} - 1)] = PT[n_{\text{clust}}(n_{\text{clust}} - 1)/2] \tag{11}$$

operations to find the *max–min* of the necessary distances in Stage 2 of the Ordering Algorithm.

*Stage 3:* In Stage 3 we need to find the minimum of $n_{\text{clust}}$ distances of each one of the remaining $(PT - n_{\text{clust}})$ training input patterns from the $n_{\text{clust}}$ cluster centers. This requires

$$(PT - n_{\text{clust}})(n_{\text{clust}} - 1) \tag{12}$$

operations, or approximately (if $n_{\text{clust}} \ll PT$)

$$PT\, n_{\text{clust}} \tag{13}$$

operations. Then, we need to sort these $(PT - n_{\text{clust}})$ minimum distances. We know that these distances range in the interval $[0, M_a]$. We first make these distances integers by multiplying each one of them by an appropriate integer $N$ (e.g., $N = 10$). This puts the distances in the range $[0, M_a N]$. Subsequently we sort these integers using radix sorting [18]. This procedure consists of the following steps.

1) Initialize $M_a N$ empty queues, one for each integer in the range one to $M_a N$.

2) Scan the sequence of integer distances $d_1, d_2, \cdots d_{\mathrm{PT}-n_{\mathrm{clust}}}$ from left to right, placing element $d_i$ in the $d_i$ queue.
3) Concatenate the queues with nonzero contents to obtain the sorted sequence.

Multiplying the $(PT - n_{\mathrm{clust}})$ distances by $N$ requires $(PT-n_{\mathrm{clust}})$ operations. We assume that it takes one operation to insert an element into the $i$th queue. So to place $(PT-n_{\mathrm{clust}})$ elements we need $(PT-n_{\mathrm{clust}})$ operations. Concatenating the $NM_a$ queues requires $NM_a$ operations. Hence, for the sorting in Stage 3, we need

$$2(PT - n_{\mathrm{clust}}) + M_a N \tag{14}$$

operations, or approximately (for $n_{\mathrm{clust}} \ll PT$)

$$2PT + M_a N \tag{15}$$

operations. Combining the max–min, and sorting related operations for Stages 1, 2, and 3 of the ordering algorithm we see that we need

$$\begin{aligned} &PT[n_{\mathrm{clust}}(n_{\mathrm{clust}} - 1)/2] + PT\,n_{\mathrm{clust}} + 2PT + M_a N \\ &\quad = PT[n_{\mathrm{clust}}(n_{\mathrm{clust}} + 1)/2 + 2 + M_a N/PT] \end{aligned} \tag{16}$$

operations. The major operations involved in the computations associated with the training phase of fuzzy ARTMAP are: addition, minimum operation, division, and comparisons. We assume that all these operations have the same cost. For the training phase of fuzzy ARTMAP we can also break down the required operations into distance related operations (which correspond to calculations of the bottom-up inputs to all committed nodes, plus one uncommitted node, in $F_2^a$), and operations related to finding the maximum of these distances. Furthermore, for fuzzy ARTMAP we have to consider the operations required to calculate weight changes as well; for simplicity of presentation we lump these weight-change related operations with the distance related equations.

*Distance Related Operations for Fuzzy ARTMAP:* Let us assume that the average number of nodes for which bottom–up inputs need to be computed in epoch 1 of fuzzy ARTMAP training is equal to $n_1$. Recall that the bottom–up input to node $j$ in $F_2^a$ of fuzzy ARTMAP is given by

$$T_j^a = \frac{|\mathbf{I} \wedge \mathbf{w}_j|}{\beta_a + |\mathbf{w}_j|} \tag{17}$$

where $\wedge$ denotes the minimum operator applied to two vectors. The minimum operator applied on two vectors $\mathbf{x}$ and $\mathbf{y}$ is a vector $\mathbf{z}$ with components the minimum of the corresponding components of $\mathbf{x}$ and $\mathbf{y}$. Also the notation $|\mathbf{x}|$ stands for the size of a vector $\mathbf{x}$, and the size of a vector is defined to be the sum of its components. Hence, the calculation of a single bottom-up input requires $2M_a$ minimum operations, and $4M_a$ addition operations (for simplicity we are omitting the single division operation). Since we have assumed an average of $n_1$ nodes in $F_2^a$ in the first epoch of training, we need, on the average

$$6M_a n_1 PT \tag{18}$$

operations for the calculation of the bottom-up inputs only. For the change of the weights during every pattern presentation we need $2M_a$ operations. Hence, the total number of operations for weight changes during an epoch is equal to $2M_aPT$. Thus for the first epoch average number of operations needed by fuzzy ARTMAP equals

$$3M_a PT(2n_1 + 2/3). \tag{19}$$

Assume that the average number of epochs required by fuzzy ARTMAP to converge is $E$. For epochs beyond epoch 1, similar formulas are valid for the number of operations required but the average number of categories in $F_2^a$ changes to $n_2$ for epoch 2, $n_3$ for epoch 3, and eventually $n_E$ for epoch $E$. Note that $n_E \geq \cdots \geq n_3 \geq n_2 \geq n_1$. Hence, the total average number of operations needed for distance related and weight changes calculations, until fuzzy ARTMAP converges, is equal to

$$3M_a PT\left(2/3E + 2\sum_{e=1}^{E} n_e\right). \tag{20}$$

*Operations in Fuzzy ARTMAP Related to Calculating Maximum of Distances:* In the first epoch of training in fuzzy ARTMAP we need to find the maximum of $n_1$ distances (the $T$'s) for every pattern presentation. Hence, the average number of operations required to find these maximum distances in the first epoch of training is equal to

$$PT(n_1 - 1). \tag{21}$$

Similarly we can find the number of operations required to obtain the maximum of $n_2$ distances (*PT* times) in epoch 2, the maximum of $n_3$ distances (*PT* times) in epoch 3, and eventually the maximum of $n_E$ distances (*PT* times) in epoch $E$. Overall the average number of maximum distance related operations in the training phase of fuzzy ARTMAP is equal to

$$PT\left[\sum_{e=1}^{E} n_e - E\right]. \tag{22}$$

Note that in (21) and (22) we have omitted the operations required when resets of nodes in $F_2^a$ occur.

*Cumulative Operations for the Ordering Algorithm and Fuzzy ARTMAP:* Combining the *number of operations required by the ordering algorithm* we get

$$\begin{aligned} &3M_a PT[1 + n_{\mathrm{clust}}(1 + n_{\mathrm{clust}})/2] \\ &\quad + PT[n_{\mathrm{clust}}(n_{\mathrm{clust}} + 1)/2 + 2 + M_a N/PT]. \end{aligned} \tag{23}$$

Also, combining the *average number of operations required by fuzzy ARTMAP* we get

$$3M_a PT\left(2/3E + 2\sum_{e=1}^{E} n_e\right) + PT\left[\sum_{e=1}^{E} n_e - E\right]. \tag{24}$$

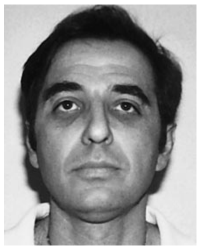
Assuming that $M_a N/PT = O(1)$, we see from the above equations that the number of operations required by the ordering algorithm and the average number of operations required by fuzzy ARTMAP are $O(PT)$. As a result, a more accurate comparison between these two algorithms should rely on the actual values of $n_{\mathrm{clust}}^2$ and $\sum_{e=1}^{E} n_e$ (see main text, second paragraph of Section V-C, for more details).

## REFERENCES

[1] P. K. Simpson, “Fuzzy min–max neural networks—Part 1: Classification,” *IEEE Trans. Neural Networks*, vol. 3, pp. 776–786, Sept. 1992.
[2] G. A. Carpenter, S. Grossberg, N. Markuzon, J. H. Reynolds, and D. B. Rosen, “Fuzzy ARTMAP: A neural-network architecture for incremental supervised learning of analog multidimensional maps,” *IEEE Trans. Neural Networks*, vol. 3, pp. 698–713, Sept. 1992.
[3] S. Grossberg, “Adaptive pattern recognition and universal recoding II: Feedback, expectation, olfaction, and illusions,” *Biol. Cybern.*, vol. 23, pp. 187–202, 1976.
[4] D. Rumelhart and J. McCleland, Eds., *Parallel Distributed Processing. Explorations in the Microstructure of Cognition*. Cambridge, MA: MIT Press, 1986.
[5] J. T. Tou and R. C. Gonzalez, *Pattern Recognition Principles*. Reading, MA: Addison-Wesley, 1976.
[6] G. A. Carpenter, S. Grossberg, and D. B. Rosen, “Fuzzy ART: Fast stable learning and categorization of analog patterns by an adaptive resonance system,” *Neural Networks*, vol. 4, pp. 759–771, 1991.
[7] M. Georgiopoulos, H. Fernlund, G. Bebis, and G. L. Heileman, “Order of search in Fuzzy ART and Fuzzy ARTMAP: Effect of the choice parameter,” *Neural Networks*, vol. 9, no. 9, pp. 1541–1559, 1996.
[8] P. Murphy and D. Ana, UCI Repository of Machine Learning databases, Dept. Comput. Sci., Univ. California, Irvine, CA, Tech. Rep., 1994. Available http://www.ics.edu/mlearn/MLRepository.html
[9] R. A. Fisher, “The use of multiple measurements in taxonomic problems,” *Annu. Eugenics*, Part II, vol. 7, pp. 179–188, 1936.
[10] S. Aeberhard, D. Coomans, and O. De Vel, “Comparison of classifiers in high-dimensional settings,” Dept. Math. Statist., James Cook Univ., North Queensland, Australia, Tech. Rep., no. 92-02, 1992.
[11] R. P. Gorman and T. J. Sejnowski, “Analysis of hidden units in a layered network trained to classify sonar targets,” *Neural Networks*, vol. 1, pp. 75–89, 1988.
[12] J. W. Smith, J. E. Everhart, W. C. Dickson, W. C. Knowler, and R. S. Johannes, “Using the ADAP learning algorithm to forecast the onset of diabetes melitus,” in *Proc. Symp. Comput. Applicat. Medical Care*, 1990, pp. 261–265.
[13] O. L. Mangasarian and W. H. Wolberg, “Cancer diagnosis via linear programming,” *SIAM News*, vol. 23, no. 5, pp. 1–18, Sept. 1990.
[14] R. S. Siegler, “Three aspects of cognitive development,” *Cognitive Psychol.*, vol. 8, pp. 481–520, 1976.
[15] A. Joshi, N. Ramakrishman, E. N. Houstis, and J. R. Rice, “On neurobiological, neuro-fuzzy, machine learning, and statistical pattern recognition techniques,” *IEEE Trans. Neural Networks*, vol. 8, pp. 18–31, 1997.
[16] M. J. Healy, T. P. Caudell, and S. D. G. Smith, “A neural-network architecture for pattern sequence verification through inferencing,” *IEEE Trans. Neural Networks*, vol. 4, pp. 9–20, 1993.
[17] G. A. Carpenter and N. Markuzon, “ARTMAP-IC and medical diagnosis: Instance counting and inconsistent cases,” *Neural Networks*, vol. 11, pp. 323–336, 1998.
[18] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, *The Design and Analysis of Computer Algorithms*. Reading, MA: Addison-Wesley, 1974.

**Issam Dagher** received the B.S. and M.S. degrees in electrical engineering from the Florida International University, Miami, in 1992 and 1994, respectively. He received the Ph.D. degree in the Electrical and Computer Engineering Department from the University of Central Florida, Orlando, in 1997.

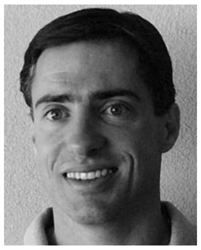
He is currently an Assistant Professor in the College of Engineering at the University of Balamand, Elkoura, Lebanon. His research interests include neural networks, fuzzy logic, and pattern recognition.



**Michael Georgiopoulos** (S’82–M’83) received the Diploma degree in electrical engineering from the National Technical University of Athens, Greece, in 1981. He also received the M.S. and Ph.D. degrees in electrical engineering from the University of Connecticut, Storrs, in 1983 and 1986, respectively.

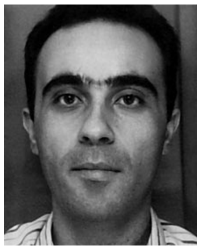
In 1987, he joined the University of Central Florida, Orlando, where he is currently an Associate Professor in the Department of Electrical and Computer Engineering. His research interests include neural networks, smart antennas, applications of neural networks in communications, pattern recognition, and fuzzy logic. He has published more than 120 technical papers in refereed journals and conferences.

Dr. Georgiopoulos is a member of the Technical Chamber in Greece and a member of the International Neural Network Society.



**Gregory L. Heileman** (S’84–M’89–SM’95) received the B.A. degree from the Wake Forest University, Winston-Salem, NC, in 1982, the M.S. degree in biomedical engineering and mathematics from the University of North Carolina, Chapel Hill, in 1986, and the Ph.D. degree in computer engineering from the University of Central Florida, Orlando, in 1989.

During 1998 he held a research fellowship at the Universidad Carlos III de Madrid, Spain. In 1990 he joined the Department of Electrical and Computer Engineering at the University of New Mexico, Albuquerque, NM, where he is an Associate Professor. His research interests include the theory of computing and information, machine learning and neural networks, and data structures and algorithmic analysis. He is the author of the text *Data Structures, Algorithms, and Object-Oriented Programming* (New York: McGraw-Hill, 1996).



**George Bebis** (S’89–M’98) received the B.S. degree in mathematics and the M.S. degree in computer science from the University of Crete, Greece, in 1987 and 1991, respectively. He received the Ph.D. degree in electrical and computer engineering from the University of Central Florida, Orlando, in 1996.

From 1996 until 1997 he was a Visiting Assistant Professor in the Department of Mathematics and Computer Science at the University of Missouri, St. Louis. From June 1998 to August 1998 he was a Summer faculty in the Center for Applied and Scientific Research (CASC) at Lawrence Livermore National Laboratory (LLNL). Currently, he is an Assistant Professor in the Department of Computer Science at the University of Nevada, Reno (UNR) and Director of the Computer Vision and Robotics Laboratory (CVRL) at UNR. His research interests include include computer vision, image processing, artificial neural networks, and genetic algorithms.

Dr. Bebis has served on the program committees of several national and international conferences and has organized and chaired several conference sessions. He is a member of the IEEE Computer Society and the Vice chair of the IEEE Northern Nevada Section.