CS 474/674 Image Processing & Interp.

Fall 2008 - Hw 2

Solutions

1) $\mathcal{F}(e^{-\pi x^2}) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi ux}\, dx =$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + j2ux)}\, dx =$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + j2ux + (ju)^2 - (ju)^2)}\, dx =$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x+ju)^2} e^{-\pi u^2}\, dx = e^{-\pi u^2} \int_{-\infty}^{\infty} e^{-\pi (x+ju)^2}\, dx$$

(set $x + ju = t \Rightarrow dx = dt$)

$$= e^{-\pi u^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi t^2}\, dt}_{1} = e^{-\pi u^2}$$

(2)

Problem 4.18

We consider the 1-D case first. From Eq. (4.4-6),

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}.$$

When $f(x)=1$ and $u=0$, $F(u)=1$. When $f(x)=1$ and $u\neq 0$,

$$F(u)=\sum_{x=0}^{M-1} e^{-j2\pi ux/M}.$$

[1, M-1]

This expression is 0 for any integer value of u in the range ~~[0, N−1]~~. There are various ways of proving this. One of the most intuitive is based on using Euler's formula to express the exponential term as

$$e^{-j2\pi ux/M}=\cos(2\pi ux/M)-j\sin(2\pi ux/M).$$

This expression describes a unit vector in the complex plane. The vector is centered at the origin and its direction depends on the value of the argument. For any integer value of u in the range ~~[0, N−1]~~ [1, M-1] the argument ranges over integer values of x in the same range. This means that the vector makes an integer number of revolutions about the origin in equal increments. Thus, for any positive value of $\cos(2\pi ux/M)$, there will be a corresponding negative value of this term. This produces a zero sum for the real part of the exponent. Similar comments apply the imaginary part. Therefore, when $f(x)=0$ and $u\neq 0$, it follows that $F(u)=0$. Thus, we have shown that for discrete quantities,

$$\Im[1]=\delta(u)=\begin{cases}1 & \text{if } u=0\\ 0 & \text{if } u\neq 0.\end{cases}$$

A similar procedure applies in the case of two variables, and we have that

$$\Im[1]=\delta(u,v)=\begin{cases}1 & \text{if } u=0 \text{ and } v=0\\ 0 & \text{otherwise.}\end{cases}$$

(3)

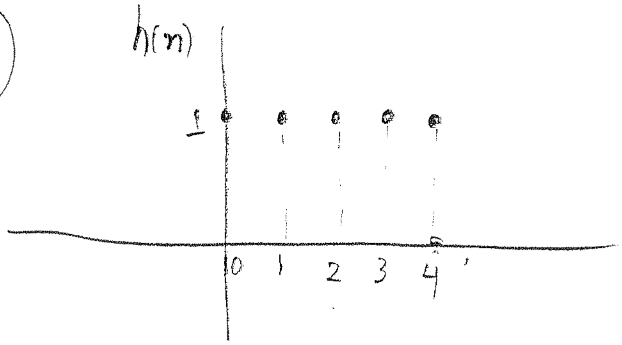
Problem 4.19

From Eq. (4.5-15), and using the exponential representation of the sine function, we have

$$\begin{aligned}
F(u,v) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\sin\left(2\pi u_0 x+2\pi v_0 y\right)\, e^{-j2\pi(ux/M+vy/N)}\\
&= \frac{-j}{2}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[e^{j2\pi(u_0x+v_0y)}-e^{-j2\pi(u_0x+v_0y)}\right]e^{-j2\pi(ux/M+vy/N)}\\
&= \frac{-j}{2}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{j2\pi(Mu_0x/M+Nv_0y/N)}e^{-j2\pi(ux/M+vy/N)}\\
&\quad -\frac{-j}{2}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}e^{-j2\pi(Mu_0x/M+Nv_0y/N)}e^{-j2\pi(ux/M+vy/N)}\\
&= \frac{-j}{2}\Im\left[(1)e^{j2\pi(Mu_0x/M+Nv_0y/N)}\right]+\frac{j}{2}\Im\left[(1)e^{-j2\pi(u_0x+v_0y)}\right]\\
&= \frac{j}{2}\left[\delta(u+Mu_0,v+Nv_0)-\delta(u-Mu_0,v-Nv_0)\right]
\end{aligned}$$

where the fourth step follows from the discussion in Problem 4.17, and the last line follows from Table 4.3.

4)



$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-j2\pi u x}{N}}, \quad u = 0, 1, \ldots, 4$$

$\underline{u=0}$ $F(0) = \frac{1}{5}\left[f(0)e^0 + f(1)e^0 + f(2)e^0 + f(3)e^0 + f(4)e^0\right] = \frac{1}{5}\left[1+1+1+1+1\right] = 1$

$\underline{u=1}$ $F(1) = \frac{1}{5}\left[f(0)e^0 + f(1)e^{\frac{-j2\pi.1}{5}} + f(2)e^{\frac{-j2\pi.2}{5}} + f(3)e^{\frac{-j2\pi.3}{5}} + f(4)e^{\frac{-j2\pi.4}{5}}\right]$

$= \frac{1}{5}\left[1 + e^{-j2\pi/5} + e^{-j4\pi/5} + e^{-j6\pi/5} + e^{-j8\pi/5}\right] =$

$= \frac{1}{5}\left[1 + (\cos 2\pi/5 - j\sin 2\pi/5) + (\cos 4\pi/5 - j\sin 4\pi/5) + (\cos 6\pi/5 - j\sin 6\pi/5) +\right.$

$\left. + (\cos 8\pi/5 - j\sin 8\pi/5)\right] = 0$

$\underline{u=2}$ $F(2) = \frac{1}{5}\left[f(0)e^0 + f(1)e^{\frac{-j2\pi.2.1}{5}} + f(2)e^{-j\frac{2\pi.2.2}{5}} + f(3)e^{-j\frac{2\pi.2.3}{5}} +\right.$

$\left. + f(4)e^{-j\frac{2\pi.2.4}{5}}\right] = \frac{1}{5}\left[1 + e^{-j4\pi/5} + e^{-j8\pi/5} + e^{-j12\pi/5} + e^{-j16\pi/5}\right] = 0$

$\underline{u=3}$ $F(3) = \frac{1}{5}\left[1 + e^{-j6\pi/5} + e^{-j12\pi/5} + e^{-j18\pi/5} + e^{-j24\pi/5}\right] = 0$

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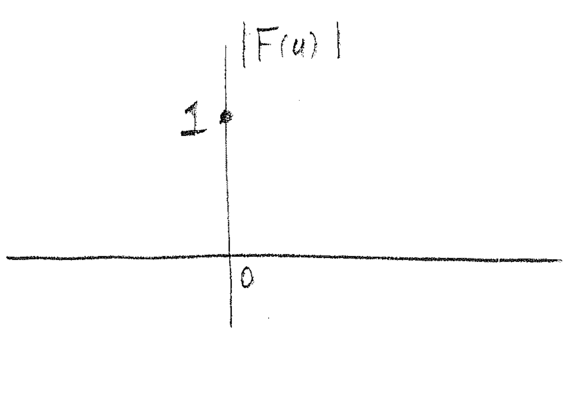
$\underline{u=4}$ $F(4) = \frac{1}{5}\left[1 + e^{-j8\pi/5} + e^{-j16\pi/5} + e^{-j24\pi/5} + e^{-j32\pi/5}\right] = 0$



(Just an <u>impulse</u>!!)

(Grad Students Only)

5) a) $f(x) = f_{even}(x) + f_{odd}(x)$ (1)

also, $f(-x) = f_{even}(-x) + f_{odd}(-x) = f_{even}(x) - f_{odd}(x)$ (2)

(1) + (2) $f(x) + f(-x) = 2 f_{even}(x) \Rightarrow f_{even} = \frac{1}{2}[f(x) + f(-x)]$

(1) − (2) $f(x) - f(-x) = 2 f_{odd}(x) \Rightarrow f_{odd} = \frac{1}{2}[f(x) - f(-x)]$

b) find $\mathcal{F}(f(-x))$ first

$\mathcal{F}(f(-x)) = \int_{-\infty}^{\infty} f(-x) e^{-j2\pi ux} dx =$ set $t = -x \Rightarrow dt = -dx$ or $\underline{dx = -dt}$

$= -\int_{\infty}^{-\infty} f(t) e^{j2\pi ut} dt = \int_{-\infty}^{\infty} f(t) e^{j2\pi ut} dt = F^*(u)$

thus, $\mathcal{F}(f_{even}) = \frac{1}{2}[\mathcal{F}(f(x)) + \mathcal{F}(f(-x))] = \frac{1}{2}[F(u) + F^*(u)] = Re(F(u))$

$\mathcal{F}(f_{odd}) = \frac{1}{2}[\mathcal{F}(f(x)) - \mathcal{F}(f(-x))] = \frac{1}{2}[F(u) - F^*(u)] = j\, Im(F(u))$