CS474/674 Image Processing and Interpretation

Fall 2009 – Dr. George Bebis

Homework 4 - Solutions

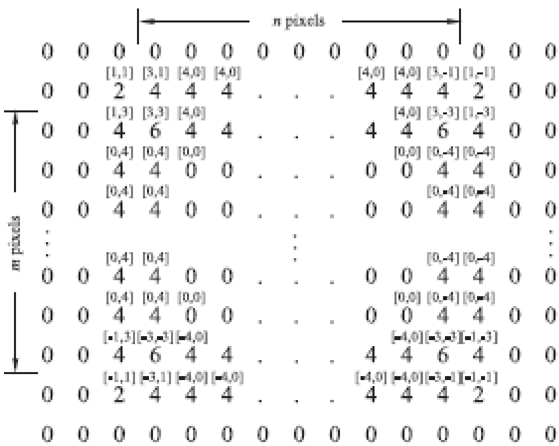
1. We have mentioned in class several times that computing convolution in the time domain is expensive; however, it can be compute more efficiently in the frequency domain using the convolution theorem and Fast Fourier Transforms (FFTs). Let's verify this claim. **(a)** What is the complexity of convolving an N x N image with an M x M mask? Use big-O notation. **(b)** What is the complexity of 2D FFT? (*hint:* use the separability property of the FT) **(c)** How could we compute the convolution using FFTs? **(d)** What is the complexity of computing the convolution in the frequency domain? Use big-O notation.

2. Problem 10.12 (page 788)

(**a**) The solution is shown in Fig. P10.12(a). The numbers in brackets are values of $[g_x, g_y]$.

(**b**) The solution is shown in Fig. P10.12(b). The angle was not computed for the trivial cases in which $g_x = g_y = 0$. The histogram follows directly from this table.

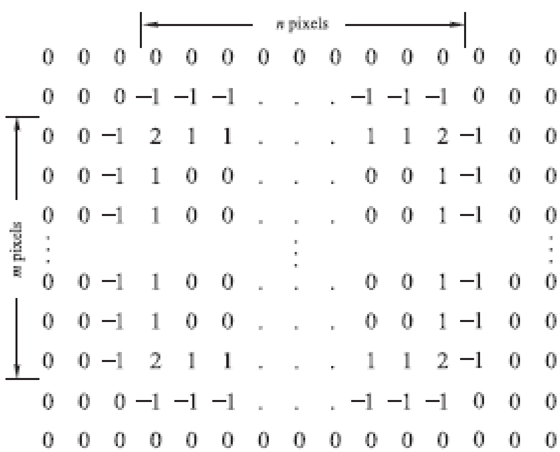
(**c**) The solution is shown in Fig. P10.12(c).



(a)

Angle	No. of points with that angle
0°	$2(n-2)$
18.4°	2
45°	4
71.6°	2
90°	$(m-2)$
−18.4°	2
−71.6°	2
−45°	4
−90°	$(n-2)$

(b)



(c)

Figure P10.12

3. Problem 10.18 (page 790)

(**a**) Equation (10.2-21) can be written in the following separable form

$$\begin{aligned} G(x,y) &= e^{-\frac{x^2+y^2}{2\sigma^2}} \\ &= e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \\ &= G(x)G(y). \end{aligned}$$

From Eq. (3.4-2) and the preceding equation, the convolution of $G(x,y)$ and $f(x,y)$ can be written as

$$\begin{aligned} G(x,y)\bigstar f(x,y) &= \sum_{s=-a}^{a}\sum_{t=-a}^{a} G(s,t)f(x-s,y-t) \\ &= \sum_{s=-a}^{a}\sum_{t=-a}^{a} e^{-\frac{s^2}{2\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} f(x-s,y-t) \\ &= \sum_{s=-a}^{a} e^{-\frac{s^2}{2\sigma^2}} \left[\sum_{t=-a}^{a} e^{-\frac{t^2}{2\sigma^2}} f(x-s,y-t)\right] \end{aligned}$$

where $a=(n-1)/2$ and n is the size of the $n\times n$ mask obtained by sampling Eq. (10.2-21). The expression inside the brackets is the 1-D convolution of the exponential term, $e^{-t^2/2\sigma^2}$, with the rows of $f(x,y)$. Then the outer summation is the convolution of $e^{-s^2/2\sigma^2}$ with the columns of the result. Stated another way,

$$G(x,y)\bigstar f(x,y) = G(x)\bigstar\left[G(y)\bigstar f(x,y)\right].$$

(**b**) Direct implementation of 2-D convolution requires n^2 multiplications at each location of $f(x,y)$, so the total number of multiplications is $n^2\times M\times N$. 1-D convolution requires n multiplications at each location of every row in the image, for a total of $n\times M\times N$ for the pass along the rows. Then, $n\times M\times N$ multiplications are required for the pass along the columns, for a total of $2nMN$ multiplications. The computational advantage, A, is then

$$A=\frac{n^2MN}{2nMN}=\frac{n}{2}$$

which is independent of image size. For example, if $n=25$, $A=12.5$, so it takes 12.5 more multiplications to implement 2-D convolution directly than it does to implement the procedure just outlined that uses 1-D convolutions.

4. **(Graduate Students Only)** Problem 10.17 (page 789)

Taking the natural log of both sides yields,

$$\ln\left[\frac{1}{2\pi\sigma_1^2}\right] - \frac{x^2+y^2}{2\sigma_1^2} = \ln\left[\frac{1}{2\pi\sigma_2^2}\right] - \frac{x^2+y^2}{2\sigma_2^2}.$$

Combining terms,

$$\begin{aligned}(x^2+y^2)\left[\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2}\right] &= \ln\left[\frac{1}{2\pi\sigma_1^2}\right] - \ln\left[\frac{1}{2\pi\sigma_2^2}\right] \\ &= \ln\left[\frac{\sigma_1^2}{\sigma_2^2}\right].\end{aligned}$$

The LoG function [Eq. (10.2-23)] is zero when $x^2+y^2=2\sigma^2$. Then, from the preceding equation,

$$\sigma^2\left[\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right] = \ln\left[\frac{\sigma_1^2}{\sigma_2^2}\right].$$

Finally, solving for σ^2,

$$\sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2-\sigma_2^2}\ln\left[\frac{\sigma_1^2}{\sigma_2^2}\right]$$

which agrees with Eq. (10.2-27).

(**b**) To obtain an expression in terms of k, we let $\sigma_1 = k\sigma_2$ in the preceding equation:

$$\begin{aligned}\sigma^2 &= \frac{k^2\sigma_2^4}{k^2\sigma_2^2-\sigma_2^2}\ln\left[\frac{k^2\sigma_2^2}{\sigma_2^2}\right] \\ &= \frac{k^2}{k^2-1}\sigma_2^2\ln(k^2)\end{aligned}$$

with $k > 1$.

(**a**) From Eq. (10.2-26), the DoG function is zero when

$$\frac{1}{2\pi\sigma_1^2}e^{-\frac{x^2+y^2}{2\sigma_1^2}} = \frac{1}{2\pi\sigma_2^2}e^{-\frac{x^2+y^2}{2\sigma_2^2}}.$$