1. We have mentioned in class several times that computing convolution in the time domain is expensive; however, it can be compute more efficiently in the frequency domain using the convolution theorem and Fast Fourier Transforms (FFTs). Let's verify this claim. (a) What is the complexity of convolving an N x N image with an M x M mask? Use big-O notation. (b) What is the complexity of 2D FFT? (hint: use the separability property of the FT) (c) How could we compute the convolution using FFTs? (d) What is the complexity of computing the convolution in the frequency domain? Use big-O notation.
2. Problem 10.12 (page 788)

(a) The solution is shown in Fig. P10.12(a). The numbers in brackets are values of \([g_x, g_y]\).

(b) The solution is shown in Fig. P10.12(b). The angle was not computed for the trivial cases in which \(g_x = g_y = 0\). The histogram follows directly from this table.

(c) The solution is shown in Fig. P10.12(c).

Figure P10.12
3. Problem 10.18 (page 790)

(a) Equation (10.2-21) can be written in the following separable form

\[ G(x, y) = e^{-\frac{x^2 + y^2}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} = G(x)G(y). \]

From Eq. (3.4-2) and the preceding equation, the convolution of \( G(x, y) \) and \( f(x, y) \) can be written as

\[ G(x, y) \star f(x, y) = \sum_{s=-a}^{a} \sum_{t=-a}^{a} G(s, t) f(x-s, y-t) \]

\[ = \sum_{s=-a}^{a} \sum_{t=-a}^{a} e^{-\frac{s^2}{2\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} f(x-s, y-t) \]

\[ = \sum_{s=-a}^{a} e^{-\frac{s^2}{2\sigma^2}} \left[ \sum_{t=-a}^{a} e^{-\frac{t^2}{2\sigma^2}} f(x-s, y-t) \right] \]

where \( a = (n - 1)/2 \) and \( n \) is the size of the \( n \times n \) mask obtained by sampling Eq. (10.2-21). The expression inside the brackets is the 1-D convolution of the exponential term, \( e^{-t^2/2\sigma^2} \), with the rows of \( f(x, y) \). Then the outer summation is the convolution of \( e^{-s^2/2\sigma^2} \) with the columns of the result. Stated another way,

\[ G(x, y) \star f(x, y) = G(x) \star [G(y) \star f(x, y)]. \]

(b) Direct implementation of 2-D convolution requires \( n^2 \) multiplications at each location of \( f(x, y) \), so the total number of multiplications is \( n^2 \times M \times N \). 1-D convolution requires \( n \) multiplications at each location of every row in the image, for a total of \( n \times M \times N \) for the pass along the rows. Then, \( n \times M \times N \) multiplications are required for the pass along the columns, for a total of \( 2nMN \) multiplications. The computational advantage, \( A \), is then

\[ A = \frac{n^2MN}{2nMN} = \frac{n}{2} \]

which is independent of image size. For example, if \( n = 25 \), \( A = 12.5 \), so it takes 12.5 more multiplications to implement 2-D convolution directly than it does to implement the procedure just outlined that uses 1-D convolutions.
4. **(Graduate Students Only)** Problem 10.17 (page 789)

Taking the natural log of both sides yields,

\[
\ln \left[ \frac{1}{2\pi\sigma_1^2} - \frac{x^2 + y^2}{2\sigma_1^2} \right] = \ln \left[ \frac{1}{2\pi\sigma_2^2} - \frac{x^2 + y^2}{2\sigma_2^2} \right].
\]

Combining terms,

\[
(x^2 + y^2) \left[ \frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2} \right] = \ln \left[ \frac{1}{2\pi\sigma_1^2} \right] - \ln \left[ \frac{1}{2\pi\sigma_2^2} \right]
\]

\[
= \ln \left[ \frac{\sigma_1^2}{\sigma_2^2} \right].
\]

The LoG function [Eq. (10.2-23)] is zero when \( x^2 + y^2 = 2\sigma^2 \). Then, from the preceding equation,

\[
\sigma^2 \left[ \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2} \right] = \ln \left[ \frac{\sigma_1^2}{\sigma_2^2} \right].
\]

Finally, solving for \( \sigma^2 \),

\[
\sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 - \sigma_2^2} \ln \left[ \frac{\sigma_1^2}{\sigma_2^2} \right]
\]

which agrees with Eq. (10.2-27).

(b) To obtain an expression in terms of \( k \), we let \( \sigma_1 = k\sigma_2 \) in the preceding equation:

\[
\sigma^2 = \frac{k^2\sigma_2^4}{k^2\sigma_2^2 - \sigma_2^2} \ln \left[ \frac{k^2\sigma_2^2}{\sigma_2^2} \right]
\]

\[
= \frac{k^2}{k^2 - 1} \sigma_2^2 \ln(k^2)
\]

with \( k > 1 \).

(a) From Eq. (10.2-26), the DoG function is zero when

\[
\frac{1}{2\pi\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}} = \frac{1}{2\pi\sigma_2^2} e^{-\frac{x^2}{2\sigma_2^2}}.
\]