CS474/674 Image Processing and Interpretation

Fall 2009 – Dr. George Bebis

Homework 6 – Solutions

1. Problem 8.1 (page 623)

(**a**) A histogram equalized image (in theory) has an intensity distribution which is uniform. That is, all intensities are equally probable. Eq. (8.1-4) thus becomes

$$L_{avg} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} l(r_k)$$

where $1/2^n$ is the probability of occurrence of any intensity. Since all intensities are equally probable, there is no advantage to assigning any particular intensity fewer bits than any other. Thus, we assign each the fewest possible bits required to cover the 2^n levels. This, of course is n bits and L_{avg} becomes n bits also:

$$\begin{aligned} L_{avg} &= \frac{1}{2^n} \sum_{k=0}^{2^n-1} (n) \\ &= \frac{1}{2^n} \left(2^n\right) n \\ &= n. \end{aligned}$$

(**b**) Since spatial redundancy is associated with the geometric arrangement of the intensitess in the image, it is possible for a histogram equalized image to contain a high level of spatial redundancy - or none at all.

2. Problem 8.2 (page 623)

(**a**) A single line of raw data contains $n_1 = 2^n$ bits. The maximum run length would be 2^n and thus require n bits for representation. The starting coordinate of each run also requires n bits since it may be arbitrarily located within the 2^n pixel line. Since a run length of 0 can not occur and the run-length pair $(0,0)$ is used to signal the start of each new line - an additional $2n$ bits are required per line. Thus, the total number of bits required to code any scan line is

$$\begin{aligned} n_2 &= 2n + N_{avg}(n+n) \\ &= 2n\left(1+N_{avg}\right) \end{aligned}$$

where N_{avg} is the average number of run-length pairs on a line. To achieve some level of compression, C must be greater than 1. So,

$$\begin{aligned} C &= \frac{n_1}{n_2} \\ &= \frac{2^n}{2n\left(1+N_{avg}\right)} > 1 \end{aligned}$$

and

$$N_{avg} < \frac{2^{n-1}}{n} - 1.$$

(**b**) For $n = 10$, N_{avg} must be less than 50.2 run-length pairs per line.

3. Problem 8.8 (page 624)

(**a**) There are two unique codes.

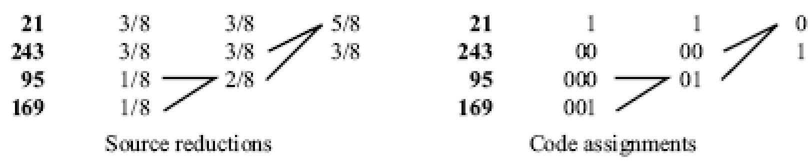

Figure P8.9

(**b**) The codes are: (1) 0, 11, 10 and (2) 1, 00, 01. The codes are complements of one another. They are constructed by following the Huffman procedure for three symbols of arbitrary probability.

4. Problem 8.9 (page 624)

(**a**) The entropy of the image is estimated using Eq. (8.1-7) to be

$$\begin{aligned} \tilde{H} &= -\sum_{k=0}^{255} p_r(r_k)\log_2 p_r(r_k) \\ &= -\left[\frac{12}{32}\log_2\frac{12}{32} + \frac{4}{32}\log_2\frac{4}{32} + \frac{4}{32}\log_2\frac{4}{32} + \frac{12}{32}\log_2\frac{12}{32}\right] \\ &= -[-0.5306 - 0.375 - 0.375 - 0.5306] \\ &= 1.811 \text{ bits/pixel.} \end{aligned}$$

The probabilities used in the computation are given in Table P8.9-1.

Table P8.9-1

Intensity	Count	Probability
21	12	3/8
95	4	1/8
169	4	1/8
243	12	3/8

(**b**) Figure P8.9 shows one possible Huffman source reduction and code assignment. Use the procedures described in Section 8.2.1. The intensities are first arranged in order of probability from the top to the bottom (at the left of the source reduction diagram). The least probable symbols are them combined to create a reduced source and the process is repeated from left to right in the diagram. Code words are then assigned to the reduced source symbols from right to left. The codes assigned to each intensity value are read from the left side of the code assignment diagram.

(**c**) Using Eq. (8.1-4), the average number of bits required to represent each pixel in the Huffman coded image (ignoring the storage of the code itself) is

$$L_{avg} = 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{15}{8} = 1.875 \text{ bits/pixel.}$$

Thus, the compression achieved is

$$C = \frac{8}{1.875} = 4.27.$$

Because the theoretical compression resulting from the elimination of all coding redundancy is $\frac{8}{1.811} = 4.417$, the Huffman coded image achieves $\frac{4.27}{4.417} \times 100$ or 96.67% of the maximum compression possible through the removal of coding redundancy alone.

(**d**) We can compute the relative frequency of pairs of pixels by assuming that the image is connected from line to line and end to beginning. The resulting probabilities are listed in Table P8.9-2.

Table P8.9-2

Intensity pair	Count	Probability
(21, 21)	8	1/4
(21, 95)	4	1/8
(95, 169)	4	1/8
(169, 243)	4	1/8
(243, 243)	8	1/4
(243, 21)	4	1/8

The entropy of the intensity pairs is estimated using Eq. (8.1-7) and dividing by 2 (because the pixels are considered in pairs):

$$
\begin{aligned}
\frac{1}{2}\tilde{H} &= -\frac{1}{2}\left[\frac{1}{4}\log_2\frac{1}{4}+\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{4}\log_2\frac{1}{4}+\frac{1}{8}\log_2\frac{1}{8}\right] \\
&= \frac{2.5}{2} \\
&= 1.25 \text{ bits/pixel.}
\end{aligned}
$$

The difference between this value and the entropy in (a) tells us that a mapping can be created to eliminate $(1.811-1.25)=0.56$ bits/pixel of spatial redundancy.

(e) Construct a difference image by replicating the first column of the original image and using the arithmetic difference between adjacent columns for the remaining elements. The difference image is

$$
\begin{matrix}
21 & 0 & 0 & 74 & 74 & 74 & 0 & 0 \\
21 & 0 & 0 & 74 & 74 & 74 & 0 & 0 \\
21 & 0 & 0 & 74 & 74 & 74 & 0 & 0 \\
21 & 0 & 0 & 74 & 74 & 74 & 0 & 0
\end{matrix}
$$

The probabilities of its various elements are given in Table 8.9-3.

Table P8.9-3

Intensity difference	Count	Probability
21	4	1/8
0	16	1/2
74	12	3/8

The entropy of the difference image is estimated using Eq. (8.1-7) to be

$$
\tilde{H} = -\left[\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{2}\log_2\frac{1}{2}+\frac{3}{8}\log_2\frac{3}{8}\right] = 1.41 \text{ bits/pixel.}
$$

(f) The entropy calculated in (a) is based on the assumption of statistically independent pixels. The entropy (of the pixel pairs) computed in (d), which is smaller that the value found in (a), reveals that the pixels are not statistically independent. There is at least $(1.811-1.25)=0.56$ bits/pixel of spatial redundancy in the image. The difference image mapping used in (e) removes most of that spatial redundancy, leaving only $(1.41-1.25)=0.16$ bits/pixel.

5. **(Graduate Students Only)** Problem 8.3 (page 623)

The original pixel intensities, their 4-bit quantized counterparts, and the differences between them are shown in Table P8.3. Note that the quantized intensities must be multiplied by 16 to decode or decompress them for the rms error and signal-to-noise calculations.

Table P8.3

$f(x,y)$		$\hat{f}(x,y)$		$16\hat{f}(x,y)-f(x,y)$
Base 10	Base 2	Base 2	Base 10	Base 10
108	01101100	0110	6	-12
139	10001011	1000	8	-11
135	10000111	1000	8	-7
244	11110100	1111	15	-4
172	10101100	1010	10	-12
173	10101101	1010	10	-13
56	00111000	0011	3	-8
99	01100011	0110	6	-3

Using Eq. (8.1-10), the rms error is

$$
\begin{aligned}
e_{rms} &= \sqrt{\frac{1}{8}\sum_{x=0}^{0}\sum_{y=0}^{7}\left[16\hat{f}(x,y)-f(x,y)\right]^2} \\
&= \sqrt{\frac{1}{8}\left[(-12)^2+(-11)^2+(-7)^2+(-4)^2+(-12)^2+(-13)^2+(-8)^2+(-3)^2\right]} \\
&= \sqrt{\frac{1}{8}(716)} \\
&= 9.46
\end{aligned}
$$

or about 9.5 intensity levels. From Eq. (8.1-11), the signal-to-noise ratio is

$$
\begin{aligned}
SNR_{ms} &= \frac{\sum_{x=0}^{0}\sum_{y=0}^{7}\left[16\hat{f}(x,y)\right]^2}{\sum_{x=0}^{0}\sum_{y=0}^{7}\left[16\hat{f}(x,y)-f(x,y)\right]^2} \\
&= \frac{96^2+128^2+128^2+240^2+160^2+160^2+48^2+96^2}{716} \\
&= \frac{162304}{716} \\
&\simeq 227.
\end{aligned}
$$