# CS474/674 Image Processing and Interpretation

# Fall 2010 – Dr. George Bebis

# Homework 7 – Solutions

1. Problem 7.9 (page 521)

**(a)** Equation (7.1-18) defines the $2 \times 2$ Haar transformation matrix as

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Thus, using Eq. (7.1-17), we get

$$\mathbf{T} = \mathbf{HFH}^T = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$
$$= \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix}$$

**(b)** First, compute

$$\mathbf{H}_2^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving this matrix equation yields

$$\mathbf{H}_2^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{H}_2 = \mathbf{H}_2^T.$$

Thus,

$$\mathbf{F} = \mathbf{H}^T\mathbf{T}\mathbf{H}$$
$$= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}.$$

2. Problem 7.12, 7.13 (page 522)

7.12

Substituting $j = 3$ into Eq. (7.2-13), we get

$$V_3 = \overline{\underset{k}{Span}\{\varphi_{3,k}(x)\}}$$
$$V_3 = \overline{\underset{k}{Span}\left\{2^{\frac{3}{2}}\varphi\left(2^3 x - k\right)\right\}}$$
$$V_3 = \overline{\underset{k}{Span}\left\{2\sqrt{2}\varphi\left(8x - k\right)\right\}}.$$

Using the Haar scaling function [Eq. (7.2-14)], we then get the result shown in Fig. P7.12.

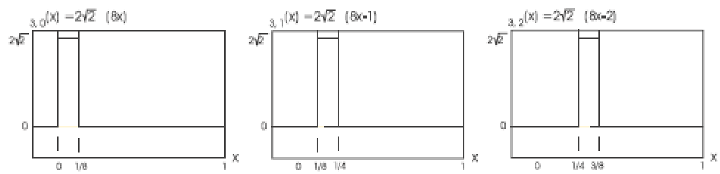



**Figure P7.12**

7.13

From Eq. (7.2-19), we find that

$$\begin{aligned}\psi_{3,3}(x) &= 2^{3/2}\psi(2^3 x - 3)\\ &= 2\sqrt{2}\psi(8x - 3)\end{aligned}$$

and using the Haar wavelet function definition from Eq. (7.2-30), obtain the plot in Fig. P7.13. To express $\psi_{3,3}(x)$ as a function of scaling functions, we employ Eq. (7.2-28) and the Haar wavelet vector defined in Example 7.6—that is, $h_\psi(0) = 1/\sqrt{2}$ and $h_\psi(1) = -1/\sqrt{2}$. Thus we get

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x - n)$$

so that

$$\begin{aligned}\psi(8x - 3) &= \sum_n h_\psi(n)\sqrt{2}\varphi(2[8x - 3] - n)\\ &= \frac{1}{\sqrt{2}}\sqrt{2}\varphi(16x - 6) + \left(\frac{-1}{\sqrt{2}}\right)\sqrt{2}\varphi(16x - 7)\\ &= \varphi(16x - 6) - \varphi(16x - 7).\end{aligned}$$

Then, since $\psi_{3,3}(x) = 2\sqrt{2}\psi(8x - 3)$ from above, substitution gives

$$\begin{aligned}\psi_{3,3} &= 2\sqrt{2}\psi(8x - 3)\\ &= 2\sqrt{2}\varphi(16x - 6) - 2\sqrt{2}\varphi(16x - 7).\end{aligned}$$

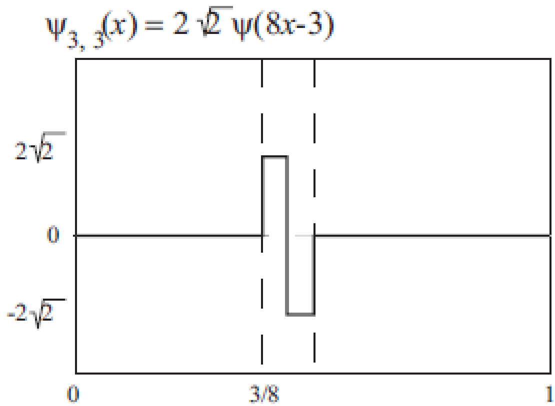



**Figure P7.13**

3. Problem 7.14 (page 522)

Using Eq. (7.2-22),

$$
\begin{aligned}
V_3 &= V_2 \oplus W_2 \\
&= V_1 \oplus W_1 \oplus W_2 \\
&= V_0 \oplus W_0 \oplus W_1 \oplus W_2
\end{aligned}
$$

The scaling and wavelet functions are plotted in Fig. P7.14.

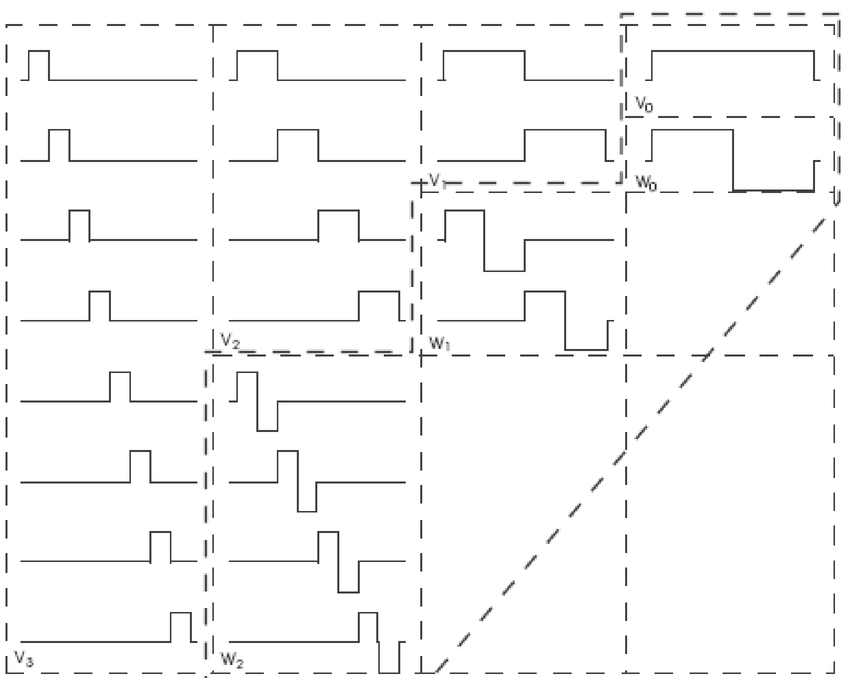



**Figure P7.14**

4. Problem 7.16, 7.19 (page 522, 523)

(**a**) Because $M = 4$, $J = 2$, and $j_0 = 1$, the summations in Eqs. (7.3-5) through (7.3-7) are performed over $n = 0,1,2,3$, $j = 1$, and $k = 0,1$. Using Haar functions and assuming that they are distributed over the range of the input sequence, we get

$$\begin{aligned}
W_\varphi(1,0) &= \frac{1}{2}\left[f(0)\varphi_{1,0}(0)+f(1)\varphi_{1,0}(1)+f(2)\varphi_{1,0}(2)+f(3)\varphi_{1,0}(3)\right] \\
&= \frac{1}{2}\left[(1)(\sqrt{2})+(4)(\sqrt{2})+(-3)(0)+(0)(0)\right] = \frac{5\sqrt{2}}{2} \\
W_\varphi(1,1) &= \frac{1}{2}\left[f(0)\varphi_{1,1}(0)+f(1)\varphi_{1,1}(1)+f(2)\varphi_{1,1}(2)+f(3)\varphi_{1,1}(3)\right] \\
&= \frac{1}{2}\left[(1)(0)+(4)(0)+(-3)(\sqrt{2})+(0)(\sqrt{2})\right] = \frac{-3\sqrt{2}}{2} \\
W_\psi(1,0) &= \frac{1}{2}\left[f(0)\psi_{1,0}(0)+f(1)\psi_{1,0}(1)+f(2)\psi_{1,0}(2)+f(3)\psi_{1,0}(3)\right] \\
&= \frac{1}{2}\left[(1)(\sqrt{2})+(4)(-\sqrt{2})+(-3)(0)+(0)(0)\right] = \frac{-3\sqrt{2}}{2} \\
W_\psi(1,1) &= \frac{1}{2}\left[f(0)\psi_{1,1}(0)+f(1)\psi_{1,1}(1)+f(2)\psi_{1,1}(2)+f(3)\psi_{1,1}(3)\right] \\
&= \frac{1}{2}\left[(1)(0)+(4)(0)+(-3)(\sqrt{2})+(0)(-\sqrt{2})\right] = \frac{-3\sqrt{2}}{2}
\end{aligned}$$

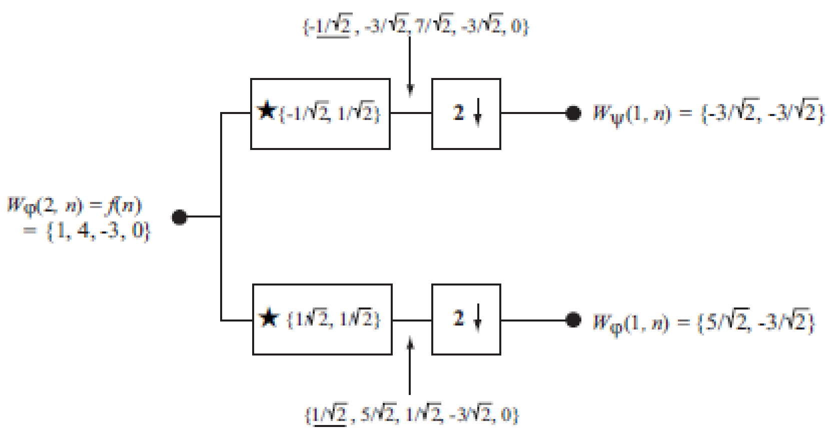
so that the DWT is $\{5\sqrt{2}/2, -3\sqrt{2}/2, -3\sqrt{2}/2, -3\sqrt{2}/2\}$.

(**b**) Using Eq. (7.3-7),

$$\begin{aligned}
f(n) = \frac{1}{2}[&W_\varphi(1,0)\varphi_{1,0}(n)+W_\varphi(1,1)\varphi_{1,1}(n)+ \\
&W_\psi(1,0)\psi_{1,0}(n)+W_\psi(1,1)\psi_{1,1}(n)]
\end{aligned}$$

which, with $n = 1$, becomes

$$\begin{aligned}
f(1) &= \frac{\sqrt{2}}{4}\left[(5)(\sqrt{2})+(-3)(0)+(-3)(\sqrt{2})+(-3)(0)\right] \\
&= \frac{2(\sqrt{2})^2}{4} = 1.
\end{aligned}$$



5. Problem 7.21 (page 523)

**(a)** Input $\varphi(n) = \{1,1,1,1,1,1,1,1\} = \varphi_{0,0}(n)$ for a three-scale wavelet transform with Haar scaling and wavelet functions. Since wavelet transform coefficients measure the similarity of the input to the basis functions, the resulting transform is

$$\{W_\varphi(0,0), W_\psi(0,0), W_\psi(1,0), W_\psi(1,1), W_\psi(2,0), W_\psi(2,1), W_\psi(2,2)$$
$$W_\psi(2,3)\} = \{2\sqrt{2}, 0, 0, 0, 0, 0, 0, 0\}.$$

The $W_\varphi(0,0)$ term can be computed using Eq. (7.3-5) with $j_0 = k = 0$.

**(b)** Using the same reasoning as in part (a), the transform is $\{0, 2\sqrt{2}, 0, 0, 0, 0, 0, 0\}$.

**(c)** For the given transform, $W_\psi(2,2) = B$ and all other transform coefficients are 0. Thus, the input must be proportional to $\psi_{2,2}(x)$. The input sequence must be of the form $\{0, 0, 0, 0, C, -C, 0, 0\}$ for some $C$. To determine $C$, use Eq. (7.3-6) to write

$$\begin{aligned}
W_\psi(2,2) = {} & \frac{1}{\sqrt{8}}\{f(0)\psi_{2,2}(0) + f(1)\psi_{2,2}(1) + f(2)\psi_{2,2}(2) + f(3)\psi_{2,2}(3) + \\
& f(4)\psi_{2,2}(4) + f(5)\psi_{2,2}(5) + f(6)\psi_{2,2}(6) + f(7)\psi_{2,2}(7)\} \\
& = \frac{1}{\sqrt{8}}\{(0)(0) + (0)(0) + (0)(0) + (0)(0) + (C)(2) + (-C)(-2) + \\
& (0)(0) + (0)(0)\} \\
& = \frac{1}{\sqrt{8}}\{2C + 2C\} = \frac{4C}{\sqrt{8}} = \sqrt{2}C.
\end{aligned}$$

Because this coefficient is known to have the value $B$, we have that $\sqrt{2}C = B$ or

$$C = \frac{\sqrt{2}}{2}B.$$

Thus, the input sequence is $\{0, 0, 0, 0, \sqrt{2}B/2, -\sqrt{2}B/2, 0, 0\}$. To check the result substitute these values into Eq. (7.3-6):

$$\begin{aligned}
W_\psi(2,2) = {} & \frac{1}{\sqrt{8}}\{(0)(0) + (0)(0) + (0)(0) + (0)(0) + (\frac{\sqrt{2}}{2}B)(2) + \\
& (-\frac{\sqrt{2}}{2}B)(-2) + (0)(0) + (0)(0)\} \\
& = \frac{1}{\sqrt{8}}\{\sqrt{2}B + \sqrt{2}B\} \\
& = B.
\end{aligned}$$

6. Problem 7.24 (page 523)

As can be seen in the sequence of images that are shown, the DWT is not shift invariant. If the input is shifted, the transform changes. Since all original images in the problem are $128 \times 128$, they become the $W_\varphi(7, m, n)$ inputs for the FWT computation process. The filter bank of Fig. 7.24(a) can be used with $j + 1 = 7$.

For a single scale transform, transform coefficients $W_\varphi(6, m, n)$ and $W_\psi^i(6, m, n)$ for $i = H, V, D$ are generated. With Haar wavelets, the transformation process subdivides the image into non-overlapping $2 \times 2$ blocks and computes 2-point averages and differences (per the scaling and wavelet vectors). Thus, there are no horizontal, vertical, or diagonal detail coefficients in the first two transforms shown; the input images are constant in all $2 \times 2$ blocks (so all differences are 0). If the original image is shifted by one pixel, detail coefficients are generated since there are then $2 \times 2$ areas that are not constant. This is the case in the third transform shown.