Name: ____________________________________________________________

1. [25 points] True/False Questions – To get credit, you must give brief reasons for each answer!

- F The filter shown below is a smoothing filter.  
  
  \[
  \begin{array}{cccc}
  1 & 2 & 1 \\
  2 & 1 & 2 \\
  1 & 2 & 1 \\
  \end{array}
  \]
  
  No negative weights.

- F Assuming an N x N image, the complexity of 2D FFT is \( O(N^2 \log N) \).

  \[
  \text{1D FFT: } N \log(N) \left\{ \begin{array}{l}
  \text{overall} \\
  \text{rows: } N^2 \log(N) \\
  \text{columns: } N^2 \log(N) \\
  \end{array} \right.
  \]

- T F The magnitude of the FT carries more information than its phase.

  The reconstruction using phase preserves more information.

- T F The Nyquist theorem holds true for band-limited functions only.

  \[
  \text{The max frequency } \omega \text{ is finite}
  \]

- F Unsharp masking is a special case of high boost filtering.

  \[
  g(x,y) = \frac{A}{A-1} f(x,y) + (f(x,y) - \frac{1}{A} f(x,y))
  \]

  Unsharp masking
2. [15 points] State and prove the convolution theorem in the continuous case. For simplicity, assume 1-D functions.

See notes
3. [15 points] Find and plot the discrete convolution of the following discrete sequences:

\[ x(n) \]

\[ h(n) \]

4 samples

2 samples
4. [20 points]. A 3 bits/pixel image of size 5x5 is given below. Find the following: (a) the output of a 3x3 averaging filter at (1,1), (b) the output of a 3x3 median filter at (1,1) and (c) the gradient magnitude at (1,1) using the Sobel masks shown below.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
y & x= & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 3 & 7 & 6 & 2 & 0 &  \\
1 & 2 & 4 & 6 & 1 & 1 &  \\
2 & 4 & 7 & 2 & 5 & 4 &  \\
3 & 3 & 0 & 6 & 2 & 1 &  \\
4 & 5 & 7 & 5 & 1 & 2 &  \\
\hline
\end{tabular}
\end{table}

\begin{align*}
\text{(a)} \quad & \frac{3+7+6+2+4+6+4+2+2}{9} = \frac{41}{9} \approx 4.56 \approx 5 \\
\text{(b)} \quad & 2 2 3 4 \, \, \, \, \, \, \, \, \, 6 \, \, \, \, \, \, \, \, \, 6 \, \, \, \, \, \, \, \, \, 7 \, \, \, \, \, \, \, \, \, 7 \, \, \, \, \, \, \, \, \, \to \, \, \, \, \, \, \, \, \, 4 \\
\text{(c)} \quad & \begin{pmatrix} 6 \end{pmatrix} = (-3-14-6) + (4+6+4) = -3 \\
& \begin{pmatrix} 6 \end{pmatrix} = (-3-14-4) + (6+12+4) = 9 \\
\text{Gradient magnitude} & = \sqrt{G_x^2 + G_y^2} = \sqrt{(-3)^2 + 9^2} = \sqrt{90} \approx 9.48 \approx 9
\end{align*}
4. [15 points] What is the FT of \( \cos(4\pi x) + \cos(10\pi x) \)? How many samples should we obtain according to the Nyquist theorem in order to avoid aliasing?

We know that \( \cos(2\pi bx) \) has the following FT:

\[
\begin{array}{c}
\text{FT:} \\
-\nu_0 & \nu_0 \\
-\nu & \nu
\end{array}
\]

\( \cos(4\pi x) = \cos(2 \cdot 2\pi 2x) \)

\( \cos(10\pi x) = \cos(2 \cdot 5\pi 5x) \)

\[
\mathcal{F} \left( \cos(4\pi x) + \cos(10\pi x) \right) = \mathcal{F} \left( \cos(4\pi x) \right) + \mathcal{F} \left( \cos(10\pi x) \right)
\]

\[
\begin{array}{c}
\text{Max freq } \nu = 5 \text{ so } \Delta x = \frac{1}{2\nu} = \frac{1}{10} \\
\text{At least 10 samples}
\end{array}
\]
5. [10 points] Given the 3x3 image shown below, compute the histogram equalized image (assume that the gray-levels are in the range [0..7]). Show all the steps.

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
4 & 1 & 0 & 1 \\
1 & 4 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccc}
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9}
\end{array}
\]

\[
S_k = T(k) = \sum_{j=0}^{L-1} P(r_j) \
\]

\[
S_0 = \frac{1}{9} \times (L-1) = \frac{7}{9} \approx 1 \\
S_1 = \left( \frac{1}{9} + \frac{4}{9} \right) \times 7 = \frac{35}{9} \approx 4 \\
S_2 = \left( \frac{1}{9} + \frac{4}{9} + \frac{1}{9} \right) \times 7 = \frac{42}{9} = \frac{14}{3} \\
S_3 = \left( \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{1}{9} \right) \times 7 = \frac{14}{9} \approx 5 \\
S_4 = S_5 = 5 \\
S_6 = \frac{56}{9} \approx 6 \\
S_7 = 7
\]

Equalized image:

\[
\begin{array}{cccc}
5 & 4 & 4 \\
4 & 7 & 6 \\
1 & 5 & 4
\end{array}
\]
7. Graduate Students Only [10 points] The pixel intensity values of a gray level image have the probability density function \( p_r(r) \) given by \( p_r(r) = 2(1-r) \), for \( 0 \leq r \leq 1 \), and zero otherwise. It is desired to transform the gray levels of the image so that they have the probability density function \( p_z(z) = 2z \), for \( 0 \leq z \leq 1 \), and zero otherwise. Assume that \( r \) and \( z \) are continuous random variables. Find the transformation that accomplishes that.

\[
P_r(r) = \begin{cases} 2(1-r) & 0 \leq r \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad P_z(z) = \begin{cases} 2z & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
S = T(r) = \int_0^r 2(1-w) \, dw = 2 \left[ \int_0^r dw - \int_0^r w \, dw \right] = 2 \left[ \frac{w^2}{2} \right]_0^r - \frac{1}{2} \left[ w^2 \right]_0^r = 2 \left( r \frac{1}{2} - \frac{1}{2} r^2 \right) = 2r - r^2 = r(2-r)
\]

\[
V = G(z) = \int_0^z 2w \, dw = 2 \left[ \frac{w^2}{2} \right]_0^z = 2 \frac{z^2}{2} = z^2
\]

\[
\zeta(r) = G^{-1}(T(r)) = \sqrt{r(2-r)}
\]