1. [25 points] True/False Questions – To get credit, you must give brief reasons for each answer!

- The filter shown below is a smoothing filter. 
  
  \[
  \begin{array}{ccc}
  1 & 2 & 1 \\
  2 & 1 & 2 \\
  1 & 2 & 1 \\
  \end{array}
  \]
  
  No negative weights.

- Assuming an NxN image, the complexity of 2D FFT is \( \mathcal{O}(N^2 \log N) \).

  \[
  \text{1D FFT: } N \log(N) \quad \text{overall} \\
  \text{N rows: } N^2 \log(N) \quad \text{each row} \\
  \text{N columns: } N^2 \log(N) \quad \text{each column} \\
  \]

- The magnitude of the FT carries more information than its phase.

  The reconstruction using phase preserves more information.

- The Nyquist theorem holds true for band-limited functions only.

  The max frequency \( \omega_d \) finite

- Unsharp masking is a special case of high boost filtering.

  Correct, when \( k = 1 \)

  \[
  g(x, y) = f(x, y) + kg_{mask}(x, y), \quad k \geq 0
  \]
2. [15 points] State and prove the convolution theorem in the continuous case. For simplicity, assume 1-D functions.
3. [15 points] Find and plot the discrete convolution of the following discrete sequences:
4. [20 points]. A 3 bits/pixel image of size 5x5 is given below. Find the following: (a) the output of a 3x3 averaging filter at (1,1), (b) the output of a 3x3 median filter at (1,1) and (c) the gradient magnitude at (1,1) using the Sobel masks shown below.

\[ \text{Magnitude} = \sqrt{G_x^2 + G_y^2} = \sqrt{(-3)^2 + 9^2} = \sqrt{90} \approx 9.49 \approx 9 \]
4. **[15 points]** What is the FT of \(\cos(4\pi x) + \cos(10\pi x)\)? How many samples should we obtain according to the Nyquist theorem in order to avoid aliasing?

We know that \(\cos(2\pi f_0 x)\) has the following

\[
\mathcal{F}\{\cos(2\pi f_0 x)\} = \frac{1}{2} \left[ \delta(f - f_0) + \delta(f + f_0) \right]
\]

\(\cos(9\pi x) = \cos(2\pi \cdot \frac{9}{2} x)\)

\(\cos(10\pi x) = \cos(2\pi \cdot 5 x)\)

\[
\mathcal{F}\{\cos(9\pi x) + \cos(10\pi x)\} = \mathcal{F}\{\cos(9\pi x)\} + \mathcal{F}\{\cos(10\pi x)\}
\]

\[
\text{max freq } \frac{1}{2w} = 5 \quad \text{so } \Delta x \leq \frac{1}{2w} = \frac{1}{10}
\]

At least 10 samples
5. [10 points] Given the 3x3 image shown below, compute the histogram equalized image (assume that the gray-levels are in the range [0..7]). Show all the steps.

\[
S_k = T(r_k) = \sum_{j=0}^{L-1} p_r(r_j) \quad L = 8
\]

\[
S_0 = \frac{1}{9} \times (L-1) = \frac{7}{9} \approx 1
\]

\[
S_1 = \left( \frac{1}{9} + \frac{4}{9} \right) \times 7 = \frac{35}{9} \approx 4
\]

\[
S_2 = \left( \frac{1}{9} + \frac{4}{9} + \frac{1}{9} \right) \times 7 = \frac{42}{9} = 5
\]

\[
S_3 = \left( \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{1}{9} \right) \times 7 = \frac{49}{9} = 5
\]

\[
S_4 = S_5 = 5
\]

\[
S_6 = \frac{56}{9} \approx 6
\]

\[
S_7 = 7
\]

Equalized image:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 4 & 5 & 5 & 5 & 5 & 6 & 7 \\
\end{array}
\]

\[
\begin{array}{c}
5 & 4 & 4 \\
4 & 7 & 6 \\
1 & 5 & 4 \\
\end{array}
\]
7. Graduate Students Only [10 points] The pixel intensity values of a gray level image have the probability density function \( p_r(r) \) given by \( p_r(r) = 2(1-r) \), for \( 0 \leq r \leq 1 \), and zero otherwise. It is desired to transform the gray levels of the image so that they have the probability density function \( p_z(z) = 2z \), for \( 0 \leq z \leq 1 \), and zero otherwise. Assume that \( r \) and \( z \) are continuous random variables. Find the transformation that accomplishes that.

\[
P_r(r) = \begin{cases} 2(1-r) & 0 \leq r \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad P_z(z) = \begin{cases} 2z & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
S = T(r) = \int_0^r 2(1-w) \, dw = 2 \left[ \int_0^r dw - \int_0^r w \, dw \right] = 2 \left[ r \int_0^r dw - \frac{1}{2} r \int_0^r w \, dw \right] = 2 \left[ r - \frac{1}{2} r^2 \right] = 2r - r^2 = r(2-r)
\]

\[
V = G(z) = \int_0^z 2w \, dw = 2 \int_0^z w \, dw = 2 \left[ \frac{1}{2} w^2 \right]_0^z = z^2 \quad \text{and} \quad z = \sqrt{V} = \sqrt{G^{-1}(r)}
\]

\[
Q(r) = G^{-1}(T(r)) = \sqrt{r(2-r)}
\]