CS485/685 Computer Vision

Spring 2010 – Dr. George Bebis

Homework 3 - Solutions

1 Find a decomposition (i.e., write A as $P\Lambda P^{-1}$) for the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad A \overset{?}{=} P\Lambda P^{-1}$

We need to find the eigenvalues and eigenvectors of A.

$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = 0 \Rightarrow -(1-\lambda)^2 = 0 \Rightarrow \lambda = \pm 1$

if $\lambda_1 = 1$ $\quad A v_1 = \lambda_1 v_1 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

$\Rightarrow v_{11} = v_{11}$, e.g., let's choose $v_{11} = 1$

$-v_{12} = v_{12} \Rightarrow 2 v_{12} = 0 \Rightarrow v_{12} = 0$

thus, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

if $\lambda_2 = -1$ $\quad A v_2 = \lambda_2 v_2 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -1 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$

$\Rightarrow v_{21} = -v_{21} \Rightarrow 2 v_{21} = 0 \Rightarrow v_{21} = 0$

$-v_{22} = -v_{22}$, e.g., $v_{22} = 1$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus, $P = [v_1\ v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Consider the 3D point $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. What would be coordinates of the point after applying the following composite transformation: (i) rotation of 90 degrees about the x-axis, (ii) translation by $d_x = -2$, $d_y = 1$, $d_z = 1$ and, (iii) scaling by $s_x = 1$, $s_y = 2$ and $s_z = 0.5$. Show your calculations clearly.

$$R_x(90) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 & 0 \\ 0 & \sin 90 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(-2,1,1) = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S(1,2,\tfrac{1}{2}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

3. Prove that the following matrix represents a rigid transformation ($a = \dfrac{\sqrt{2}}{2}$).

$$\begin{bmatrix} a & -a & 0 \\ a & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

)
$$\begin{bmatrix} a & -a & 0 \\ a & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ where } a = \frac{\sqrt{2}}{2}$$

All we have to show is that the 2×2 sub-matrix is orthonormal.

$$\begin{bmatrix} a \\ a \end{bmatrix} \cdot \begin{bmatrix} -a \\ a \end{bmatrix} = -a^2 + a^2 = 0$$

dot product

$$\left\| \begin{bmatrix} a \\ a \end{bmatrix} \right\| = \left\| \begin{bmatrix} -a \\ a \end{bmatrix} \right\| = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2} = 1$$

4 It is easy to show that the unit vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis in R^3. Prove that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ form also a basis of R^3. What is the representation of vector $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ in this basis?

We need to show that:

(i) v_1, v_2, v_3 are linearly independent

(ii) every vector $v \in R^3$ can be written as a linear combination of v_1, v_2, v_3

(i) $\Rightarrow$ if $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \overset{?}{\Rightarrow} c_1 = c_2 = c_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow$$

$$c_1 + c_2 + c_3 = 0, \quad c_2 + c_3 = 0, \quad c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

(ii) Suppose $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow V = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} \Rightarrow \begin{matrix} c_3 = z \\ c_2 = y - z \\ c_1 = x - y \end{matrix}$$

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$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \overset{?}{=} c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

essentially, $x = y = z = 3$ in this case. From part A, $c_3 = 3$, $c_2 = 0$, $c_1 = 0$. Thus, the representation of $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ in the basis v_1, v_2, v_3 is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

Graduate Students Only

5 Suppose A is a real mxn matrix. Prove that the squares of the singular values of A are the eigenvalues of $A^T A$. (*hint*: if A is a symetric matrix, it can be written as $A = P \Lambda P^T$ where the columns of P are the eigenvectors of A and Λ is a diagonal matrix with diagonal elements equal to the eigenvalues of A).

) suppose $A = \underline{UDV^T}$ is the SVD of A.

then, $A^T = (UDV^T)^T = (V^T)^T D^T U^T = \underline{VD^TU^T}$.

Then, $A^TA = (VD^TU^T)(UDV^T) = VDD^TV^T$

$= \underline{VD^2V^T}$ the eigenvalues ~~singular values~~ of A^TA are the squares of the singular values of A.