Since λ is a scalar, it can change position on the right-hand side of the equation. Also, because of the associativity of matrix multiplication, we may write:

$$(\mathbf{u}^T g)(g^T \mathbf{u}) = \lambda \mathbf{u}^T \mathbf{u} \tag{2.43}$$

Since $\mathbf{u}$ is an eigenvector, $\mathbf{u}^T \mathbf{u} = 1$. Therefore:

$$(g^T \mathbf{u})^T (g^T \mathbf{u}) = \lambda \tag{2.44}$$

$g^T \mathbf{u}$ is some vector $\mathbf{y}$. Then we have: $\lambda = \mathbf{y}^T \mathbf{y}$ which means that λ is non-negative since $\mathbf{y}^T \mathbf{y}$ is the square magnitude of vector $\mathbf{y}$.

Example 2.7

If λ_i are the eigenvalues of gg^T and $\mathbf{u}_i$ the corresponding eigenvectors, show that $g^T g$ has the same eigenvalues, with the corresponding eigenvectors given by $\mathbf{v}_i = g^T \mathbf{u}_i$.

By definition:

$$gg^T \mathbf{u}_i = \lambda_i \mathbf{u}_i \tag{2.45}$$

Multiply both sides from the left with g^T:

$$g^T gg^T \mathbf{u}_i = g^T \lambda_i \mathbf{u}_i \tag{2.46}$$

As λ_i is a scalar, it may change position with respect to the other factors on the right-hand side of (2.46). Also, by the associativity of matrix multiplication:

$$g^T g(g^T \mathbf{u}_i) = \lambda_i (g^T \mathbf{u}_i) \tag{2.47}$$

This identifies $g^T \mathbf{u}_i$ as an eigenvector of $g^T g$ with λ_i the corresponding eigenvalue.

Example 2.8

You are given an image: $g = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. **Compute the eigenvectors $\mathbf{u}_i$ of**

$$g^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.48}$$

We start by computing first gg^T*:*

$$gg^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.49}$$

The eigenvalues of gg^T *will be computed from its characteristic equation:*

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 6-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(6-\lambda)(1-\lambda)-1] - 2[2(1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[(6-\lambda)(1-\lambda)-1-4] = 0 \tag{2.50}$$

One eigenvalue is $\lambda = 1$. *The other two are the roots of:*

$$6 - 6\lambda - \lambda + \lambda^2 - 5 = 0 \Rightarrow \lambda^2 - 7\lambda + 1 = 0 \Rightarrow \lambda = \frac{7 \pm \sqrt{49-4}}{2} = \frac{7 \pm 6.7}{2}$$

$$\Rightarrow \lambda = 6.854 \text{ or } \lambda = 0.146 \tag{2.51}$$

In descending order, the eigenvalues are:

$$\lambda_1 = 6.854,\ \lambda_2 = 1,\ \lambda_3 = 0.146 \tag{2.52}$$

Let $\mathbf{u}_i = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ *be the eigenvector which corresponds to eigenvalue* λ_i. *Then:*

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{array}{c} x_1 + 2x_2 = \lambda_i x_1 \\ 2x_1 + 6x_2 + x_3 = \lambda_i x_2 \\ x_2 + x_3 = \lambda_i x_3 \end{array} \tag{2.53}$$

For $\lambda_i = 6.854$

$$2x_2 - 5.854x_1 = 0 \tag{2.54}$$

$$2x_1 - 0.854x_2 + x_3 = 0 \tag{2.55}$$

$$x_2 - 5.854x_3 = 0 \tag{2.56}$$

Multiply (2.55) with 5.854 *and add equation (2.56) to get:*

$$11.7x_1 - 4x_2 = 0 \tag{2.57}$$

Equation (2.57) is the same as (2.54). So we have really only two independent equations for the three unknowns. We choose the value of x_1 to be 1. Then:

$$x_2 = 2.927 \text{ and from (2.55) } x_3 = -2 + 0.85 \times 2.925 = -2 + 2.5 = 0.5 \tag{2.58}$$

Thus, the first eigenvector is

$$\begin{pmatrix} 1 \\ 2.927 \\ 0.5 \end{pmatrix} \tag{2.59}$$

and after normalisation, ie division with $\sqrt{1^2 + 2.927^2 + 0.5^2} = 3.133$, we obtain:

$$\mathbf{u_1} = \begin{pmatrix} 0.319 \\ 0.934 \\ 0.160 \end{pmatrix} \tag{2.60}$$

For $\lambda_i = 1$, the system of linear equations we have to solve is:

$$\begin{aligned} x_1 + 2x_2 = x_1 &\Rightarrow x_2 = 0 \\ 2x_1 + x_3 = 0 &\Rightarrow x_3 = -2x_1 \end{aligned} \tag{2.61}$$

Choose $x_1 = 1$. Then $x_3 = -2$. Since $x_2 = 0$, we must divide all components with $\sqrt{1^2 + 2^2} = \sqrt{5}$ for the eigenvector to have unit length:

$$\mathbf{u_2} = \begin{pmatrix} 0.447 \\ 0 \\ -0.894 \end{pmatrix} \tag{2.62}$$

For $\lambda_i = 0.146$, the system of linear equations we have to solve is:

$$\begin{aligned} 0.854x_1 + 2x_2 &= 0 \\ 2x_1 + 5.854x_2 + x_3 &= 0 \\ x_2 + 0.854x_3 &= 0 \end{aligned} \tag{2.63}$$

Choose $x_1 = 1$. Then $x_2 = -\frac{0.854}{2} = -0.427$ and $x_3 = -\frac{0.427}{0.854} = 0.5$. Therefore, the third eigenvector is:

$$\begin{pmatrix} 1 \\ -0.427 \\ 0.5 \end{pmatrix}, \tag{2.64}$$

and after division with $\sqrt{1 + 0.427^2 + 0.5^2} = 1.197$ we obtain:

$$\begin{pmatrix} 0.835 \end{pmatrix}$$

The corresponding eigenvectors of $g^T g$ are given by $g^T \mathbf{u}_i$; ie the first one is:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.319 \\ 0.934 \\ 0.160 \end{pmatrix} = \begin{pmatrix} 2.187 \\ 0.934 \\ 1.094 \end{pmatrix} \quad (2.66)$$

We normalise it by dividing with $\sqrt{2.187^2 + 0.934^2 + 1.094^2} = 2.618$, *to obtain:*

$$\mathbf{v_1} = \begin{pmatrix} 0.835 \\ 0.357 \\ 0.418 \end{pmatrix} \quad (2.67)$$

Similarly

$$\mathbf{v_2} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.447 \\ 0 \\ -0.894 \end{pmatrix} = \begin{pmatrix} 0.447 \\ 0 \\ -0.894 \end{pmatrix}, \quad (2.68)$$

while the third eigenvector is

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.835 \\ -0.357 \\ 0.418 \end{pmatrix} = \begin{pmatrix} 0.121 \\ -0.357 \\ 0.061 \end{pmatrix} \quad (2.69)$$

which after normalisation becomes:

$$\mathbf{v_3} = \begin{pmatrix} 0.319 \\ -0.934 \\ 0.160 \end{pmatrix} \quad (2.70)$$

What is the singular value decomposition of an image?

The Singular Value Decomposition (SVD) of an image f is its expansion in terms of vector outer products, where the vectors used are the eigenvectors of ff^T and f^Tf, and the coefficients of the expansion are the eigenvalues of these matrices. In that case, equation (2.9) may be written as

$$f = \sum_{i=1}^{r} \lambda_i^{\frac{1}{2}} \mathbf{u_i} \mathbf{v_i}^T \quad (2.71)$$

since the only nonzero terms are those with $i = j$. Elementary images $\mathbf{u_i}\mathbf{v_i}^T$ are known as the **eigenimages** of image f.