Since $\lambda$ is a scalar, it can change position on the right-hand side of the equation. Also, because of the associativity of matrix multiplication, we may write:

$$(u^T g)(g^T u) = \lambda u^T u$$  \hspace{1cm} (2.43)$$

Since $u$ is an eigenvector, $u^T u = 1$. Therefore:

$$(g^T u)^T (g^T u) = \lambda$$  \hspace{1cm} (2.44)$$

g^T u$ is some vector $y$. Then we have: $\lambda = y^T y$ which means that $\lambda$ is non-negative since $y^T y$ is the square magnitude of vector $y$.

**Example 2.7**

If $\lambda_i$ are the eigenvalues of $g g^T$ and $u_i$ the corresponding eigenvectors, show that $g^T g$ has the same eigenvalues, with the corresponding eigenvectors given by $v_i = g^T u_i$.

*By definition:*

$gg^T u_i = \lambda_i u_i$  \hspace{1cm} (2.45)$$

*Multiply both sides from the left with $g^T$:*

$$g^T gg^T u_i = g^T \lambda_i u_i$$  \hspace{1cm} (2.46)$$

As $\lambda_i$ is a scalar, it may change position with respect to the other factors on the right-hand side of (2.46). Also, by the associativity of matrix multiplication:

$$g^T g(g^T u_i) = \lambda_i (g^T u_i)$$  \hspace{1cm} (2.47)$$

This identifies $g^T u_i$ as an eigenvector of $g^T g$ with $\lambda_i$ the corresponding eigenvalue.

**Example 2.8**

You are given an image: $g = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Compute the eigenvectors $u_i$ of
We start by computing first $gg^T$:

$$
 gg^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix}
$$

The eigenvalues of $gg^T$ will be computed from its characteristic equation:

$$
\begin{vmatrix}
1 - \lambda & 2 & 0 \\
2 & 6 - \lambda & 1 \\
0 & 1 & 1 - \lambda
\end{vmatrix} = 0 \Rightarrow (1 - \lambda)(6 - \lambda)(1 - \lambda) - 2[2(1 - \lambda)] = 0
$$

$$
\Rightarrow (1 - \lambda)(6 - \lambda)(1 - \lambda) - 1 - 4 = 0 \quad (2.50)
$$

One eigenvalue is $\lambda = 1$. The other two are the roots of:

$$
6 - 6\lambda - \lambda^2 - 5 = 0 \Rightarrow \lambda^2 - 7\lambda + 1 = 0 \Rightarrow \lambda = \frac{7 \pm \sqrt{49 - 4}}{2} = \frac{7 \pm 6.7}{2}
$$

$$
\Rightarrow \lambda = 6.854 \text{ or } \lambda = 0.146 \quad (2.51)
$$

In descending order, the eigenvalues are:

$$
\lambda_1 = 6.854, \lambda_2 = 1, \lambda_3 = 0.146
$$

Let $u_i = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector which corresponds to eigenvalue $\lambda_i$. Then:

$$
\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = \lambda_1 x_1 \\ 2x_1 + 6x_2 + x_3 = \lambda_2 x_2 \\ x_2 + x_3 = \lambda_3 x_3 \end{cases}
$$

For $\lambda_i = 6.854$

$$
2x_2 - 5.854x_1 = 0 \quad (2.54)
$$

$$
2x_1 - 0.854x_2 + x_3 = 0 \quad (2.55)
$$

$$
x_2 - 5.854x_3 = 0 \quad (2.56)
$$

Multiply (2.55) with 5.854 and add equation (2.56) to get:

$$
11.7x_1 - 4x_2 = 0 \quad (2.57)
$$
Equation (2.57) is the same as (2.54). So we have really only two independent equations for the three unknowns. We choose the value of \( x_1 \) to be 1. Then:

\[
x_2 = 2.927 \quad \text{and from (2.55) } x_3 = -2 + 0.85 \times 2.925 = -2 + 2.5 = 0.5 \quad (2.58)
\]

Thus, the first eigenvector is

\[
\begin{pmatrix}
1 \\
2.927 \\
0.5
\end{pmatrix}
\]

and after normalisation, i.e. division with \( \sqrt{1^2 + 2.927^2 + 0.5^2} = 3.133 \), we obtain:

\[
\mathbf{u}_1 = \begin{pmatrix}
0.319 \\
0.934 \\
0.160
\end{pmatrix} \quad (2.60)
\]

For \( \lambda_i = 1 \), the system of linear equations we have to solve is:

\[
x_1 + 2x_2 = x_1 \Rightarrow x_2 = 0 \\
2x_1 + x_3 = 0 \Rightarrow x_3 = -2x_1
\]

Choose \( x_1 = 1 \). Then \( x_3 = -2 \). Since \( x_2 = 0 \), we must divide all components with \( \sqrt{1^2 + 2^2} = \sqrt{5} \) for the eigenvector to have unit length:

\[
\mathbf{u}_2 = \begin{pmatrix}
0.447 \\
0 \\
-0.894
\end{pmatrix} \quad (2.62)
\]

For \( \lambda_i = 0.146 \), the system of linear equations we have to solve is:

\[
0.854x_1 + 2x_2 = 0 \\
2x_1 + 5.854x_2 + x_3 = 0 \\
x_2 + 0.854x_3 = 0
\]

Choose \( x_1 = 1 \). Then \( x_2 = -\frac{0.854}{2} = -0.427 \) and \( x_3 = \frac{-0.427}{0.854} = 0.5 \). Therefore, the third eigenvector is:

\[
\begin{pmatrix}
1 \\
-0.427 \\
0.5
\end{pmatrix}
\]

and after division with \( \sqrt{1 + 0.427^2 + 0.5^2} = 1.197 \) we obtain:

\[
\begin{pmatrix}
1 \\
-0.835 \\
0
\end{pmatrix}
\]
The corresponding eigenvectors of \( g^T g \) are given by \( g^T u_1 \); ie the first one is:

\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0.319 \\
0.934 \\
0.160
\end{pmatrix}
= \begin{pmatrix} 2.187 \\ 0.934 \\ 1.094 \end{pmatrix}
\] (2.66)

We normalise it by dividing with \( \sqrt{2.187^2 + 0.934^2 + 1.094^2} = 2.618 \), to obtain:

\[
\nu_1 = \begin{pmatrix} 0.835 \\ 0.357 \\ 0.418 \end{pmatrix}
\] (2.67)

Similarly

\[
\nu_2 = \begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0.447 \\
0 \\
-0.894
\end{pmatrix}
= \begin{pmatrix} 0.447 \\ 0 \\ -0.894 \end{pmatrix}
\] (2.68)

while the third eigenvector is

\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0.835 \\
-0.357 \\
0.418
\end{pmatrix}
= \begin{pmatrix} 0.121 \\ -0.357 \\ 0.061 \end{pmatrix}
\] (2.69)

which after normalisation becomes:

\[
\nu_3 = \begin{pmatrix} 0.319 \\ -0.934 \\ 0.160 \end{pmatrix}
\] (2.70)

What is the singular value decomposition of an image?

The Singular Value Decomposition (SVD) of an image \( f \) is its expansion in terms of vector outer products, where the vectors used are the eigenvectors of \( ff^T \) and \( f^T f \), and the coefficients of the expansion are the eigenvalues of these matrices. In that case, equation (2.9) may be written as

\[
f = \sum_{i=1}^{r} \lambda_i \frac{1}{\sqrt{\lambda_i}} u_i v_i^T
\] (2.71)

since the only nonzero terms are those with \( i = j \). Elementary images \( u_i v_i^T \) are known as the eigenimages of image \( f \).