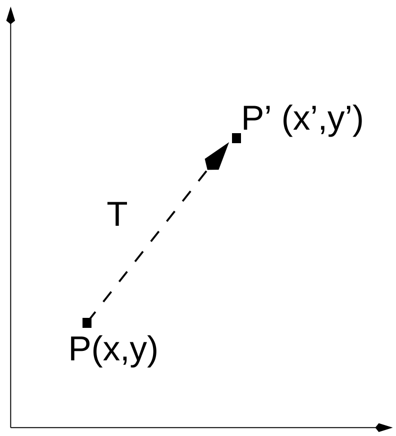
# 2D Geometrical Transformations

- **Translation**

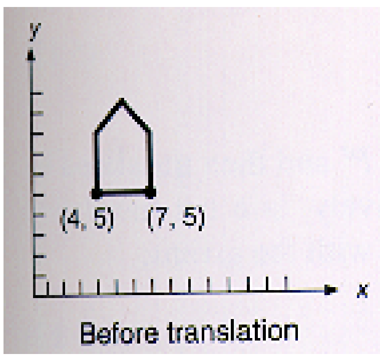
- Moves points to new locations by adding translation amounts to the coordinates of the points

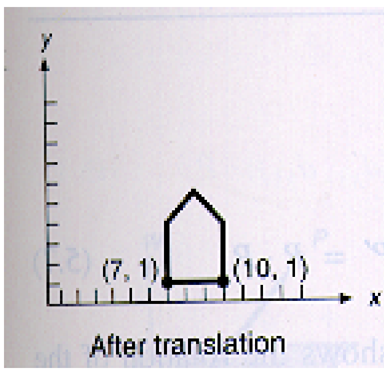



$$x' = x + dx, \quad y' = y + dy \text{ or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\underline{P' = P + T}$$

- To translate an object, translate every point of the object by the same amount

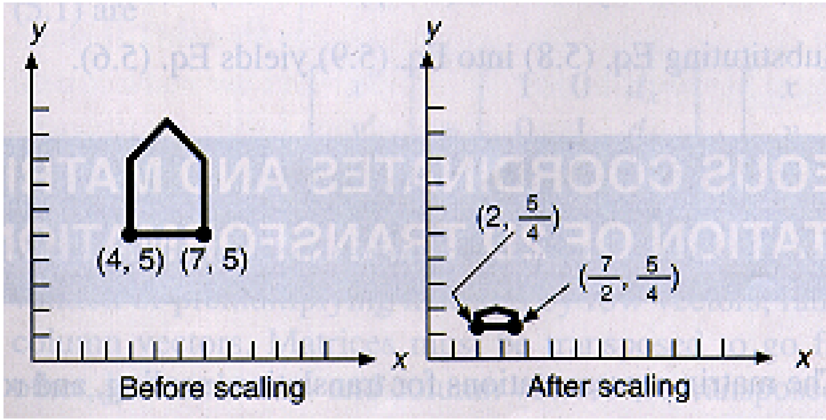



(translate only the endpoints of line segments - redrawing is required)

- **Scaling**

- Changes the size of the object by multiplying the coordinates of the points by scaling factors




$$x' = x\ s_x,\ \ y' = y\ s_y \ \text{ or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},\ \ s_x, s_y > 0$$

$$\underline{P' = S\ P}$$

- Scale factors affect size as following:

* If $s_x = s_y$ uniform scaling

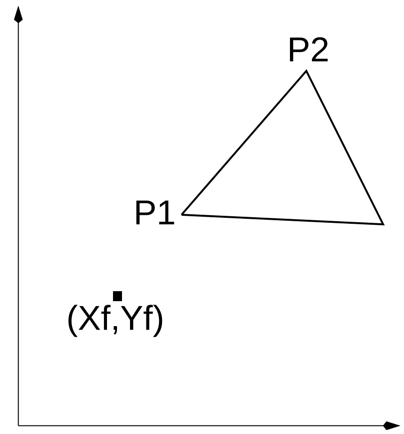
* If $s_x \neq s_y$ nonuniform scaling

* If $s_x, s_y < 1$, size is reduced, object moves closer to origin

* If $s_x, s_y > 1$, size is increased, object moves further from origin

* If $s_x = s_y = 1$, size does not change

- Control the location of a scaled object by choosing the location of a point (*fixed point*) with respect to which the scaling is performed




$$x' = x_f + (x - x_f)s_x \quad \text{or} \quad x' = xs_x + x_f(1 - s_x)$$

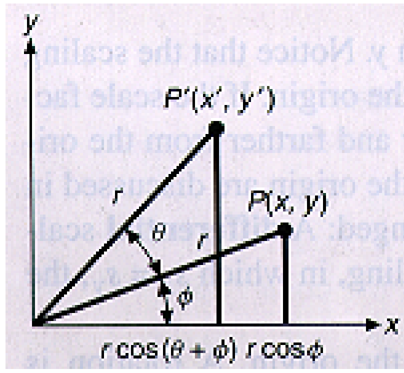
$$y' = y_f + (y - y_f)s_y \quad \text{or} \quad y' = ys_y + y_f(1 - s_y)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_f(1 - s_x) \\ y_f(1 - s_y) \end{bmatrix}$$

$$\underline{P' = S\ P + T_f}$$

- **Rotation**

  - Rotates points by an angle $\theta$ about origin ($\theta$>0: counterclockwise rotation)




  - From $ABP$ triangle:

$$cos(\phi) = x/r \text{ or } x = rcos(\phi)$$
$$sin(\phi) = y/r \text{ or } y = rsin(\phi)$$

  - From $ACP'$ triangle:

$$cos(\phi + \theta) = x'/r \text{ or } x' = rcos(\phi + \theta) = rcos(\phi)cos(\theta) - rsin(\phi)sin(\theta)$$
$$sin(\phi + \theta) = y'/r \text{ or } y' = rsin(\phi + \theta) = rcos(\phi)sin(\theta) + rsin(\phi)cos(\theta)$$

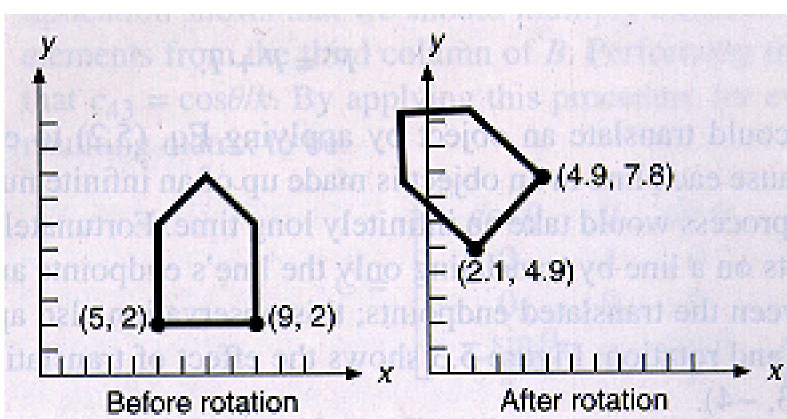
  - From the above equations we have:

$$x' = xcos(\theta) - ysin(\theta), \quad y' = xsin(\theta) + ycos(\theta) \text{ or}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

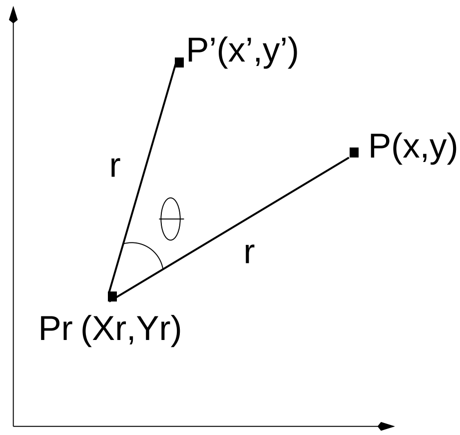
$$\underline{P' = R\ P}$$

  - To rotate an object, rotate every point of the object by the same amount



(rotate only the endpoints of line segments - redrawing is required)

- Performing rotation about an arbitrary point




$$x' = x_r + (x - x_r)cos(\theta) - (y - y_r)sin(\theta)$$

$$y' = y_r + (x - x_r)sin(\theta) + (y - y_r)cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\underline{P' = R(P - P_r) + P_r}$$

- **Summary of transformations**

Translation: $P' = P + T$ (addition causes problems !!)

Scale: $P' = S\ P$

Rotation: $P' = R\ P$

- Idea: use homogeneous coordinates to express translation as matrix multiplication

- **Homogeneous coordinates (projective space)**

- Idea: add a third coordinate: $(x, y)$ --> $(x_h, y_h, w)$

- Homogenize $(x_h, y_h, w)$:

$$x = \frac{x_h}{w}, \quad y = \frac{y_h}{w}, \; w \neq 0$$

- In general: $(x, y)$ --> $(xw, yw, w)$ (i.e., $x_h = xw$, $y_h = yw$)

- $w$ can assume any value ($w \neq 0$), for example, $w = 1$:

$(x, y) --> (x, y, 1)$ (no division is required when you homogenize !!)

$(x, y) --> (2x, 2y, 2)$ (division is required when you homogenize !!)

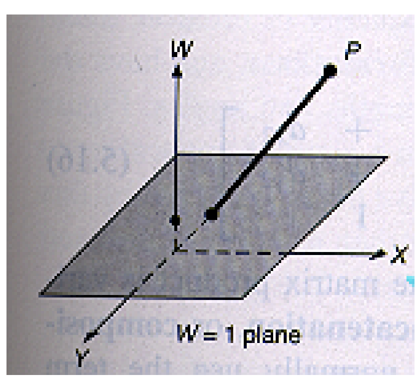
- $(x, y)$ can be represented by an infinite number of points in homogeneous coordinates

If $w = 6$, $(1/3, 1/2) --> (2, 3, 6)$

If $w = 12$, $(1/3, 1/2) --> (4, 6, 12)$

...

- All these points lie on a line in the space of homogeneous coordinates !!



- **Translation using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

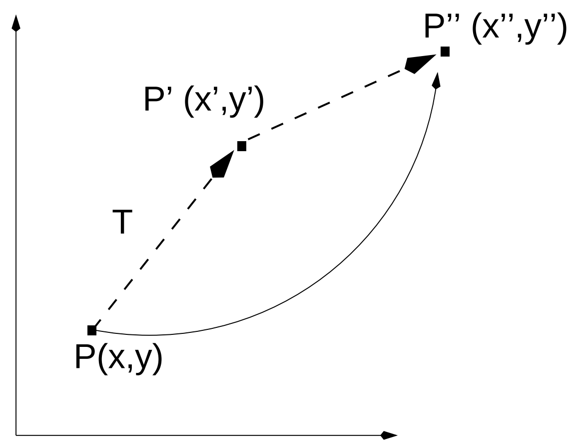
(Verification: $x' = 1x + 0y + 1dx = x + dx$)

(Verification: $y' = 0x + 1y + 1dy = y + dy$)

(Homogenize: divide by 1 !!)

$\underline{P' = T(dx, dy)\ P}$

- Successive translations




$$P' = T(dx_1, dy_1)\ P, \quad P'' = T(dx_2, dy_2)\ P'$$

Thus, $P'' = T(dx_2, dy_2)T(dx_1, dy_1)\ P = T(dx_1 + dx_2, dy_1 + dy_2)\ P$

$$\begin{bmatrix} 1 & 0 & dx_2 \\ 0 & 1 & dy_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & dx_1 \\ 0 & 1 & dy_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx_1 + dx_2 \\ 0 & 1 & dy_1 + dy_2 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Scaling using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(Verify: $x' = s_x x + 0y + 01 = s_x x$)

(Verify: $y' = 0x + s_y y + 01 = s_y y$)

(homogenize: divide by 1 !!)

$\underline{P' = S(s_x, s_y)\ P}$

- Successive scalings

$$P' = S(s_{x_1}, s_{y_1})\ P, \quad P'' = S(s_{x_2}, s_{y_2})\ P'$$

Thus, $P'' = S(s_{x_2}, s_{y_2})S(s_{x_1}, s_{y_1})\ P = S(s_{x_1}s_{x_2}, s_{y_1}s_{y_2})\ P$

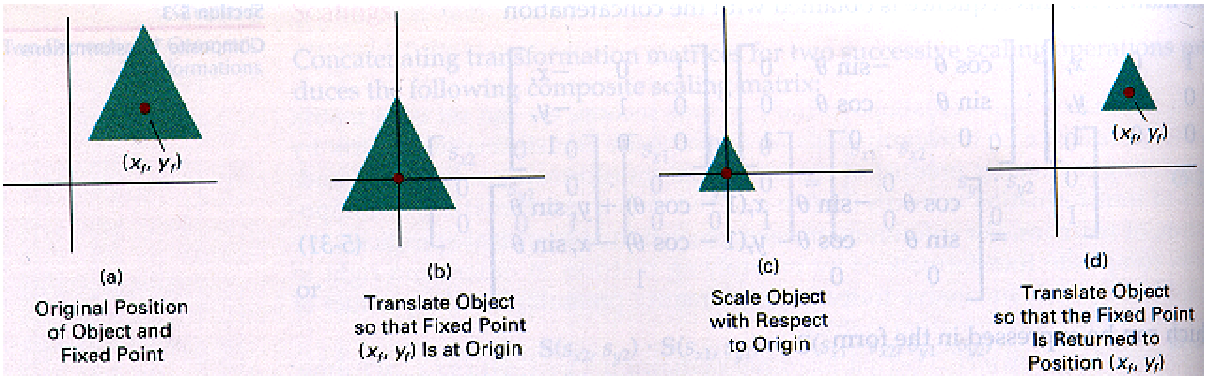
$$\begin{bmatrix} s_{x_2} & 0 & 0 \\ 0 & s_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x_1} & 0 & 0 \\ 0 & s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x_2}s_{x_1} & 0 & 0 \\ 0 & s_{y_2}s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Scaling about a fixed point

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Decomposing the above transformation:

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix}$$




- **Rotation using homogeneous coordinates**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underline{P' = R(\theta)\ P}$$

- Successive rotations

$$P' = R(\theta_1)\ P, \quad P'' = R(\theta_2)\ P'$$

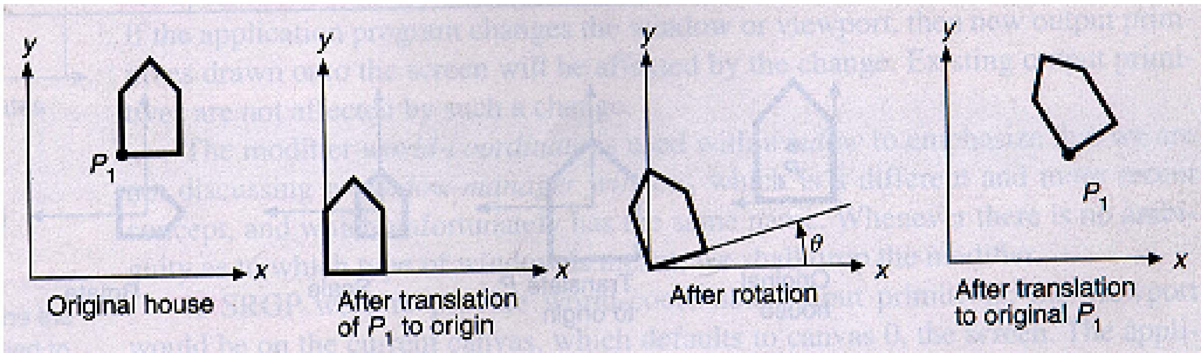
$$\text{Thus, } P'' = R(\theta_1)R(\theta_2)\ P = R(\theta_1 + \theta_2)\ P$$

- Rotation about an arbitrary point

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & x_r(1 - cos(\theta)) + y_r sin(\theta) \\ sin(\theta) & cos(\theta) & y_r(1 - cos(\theta)) - x_r sin(\theta) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Decomposing the above transformation:

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$




- **Composition of a series of transformations**

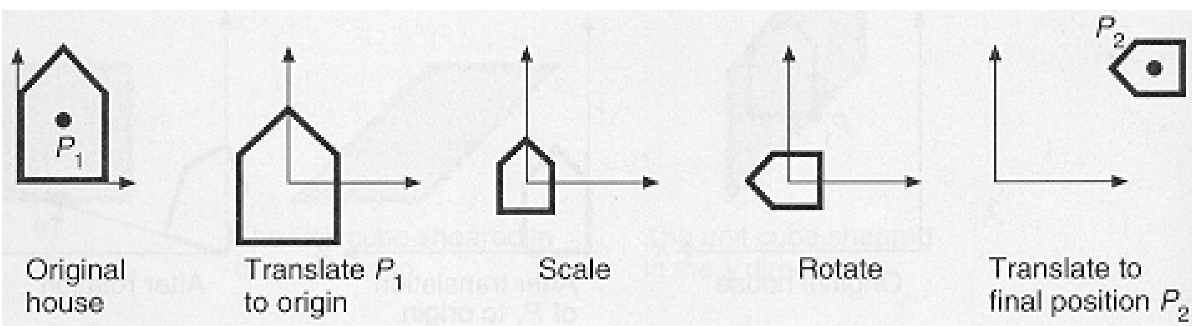
  - The transformation matrices of a series of transformations can be concatenated into a single transformation matrix

  - Example

    * Translate $P_1$ to origin

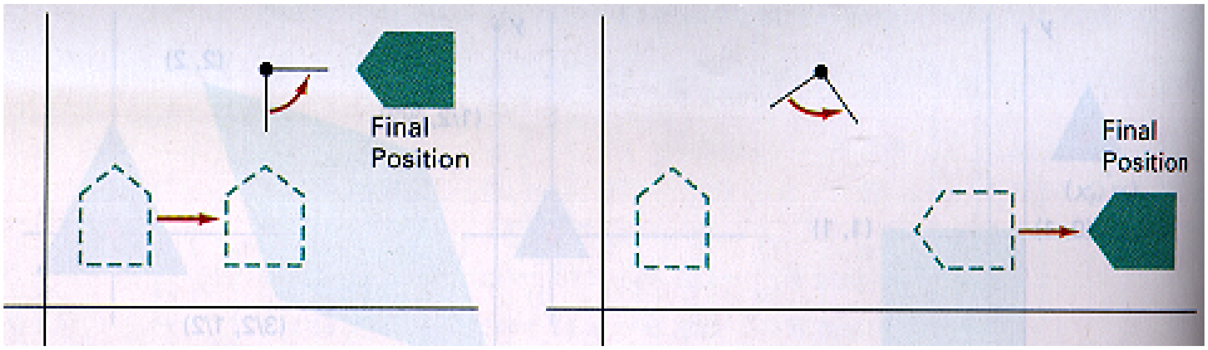
    * Perform scaling and rotation

    * Translate to $P_2$




$$M = T(x_2, y_2)R(\theta)S(s_x, s_y)T(-x_1, -y_1)$$

- It is important to reserve the order in which a sequence of transformations is performed !!




- **General form of transformation matrix**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Representing a sequence of transformations as a single transformation matrix is more efficient

$$x' = a_{11}x + a_{12}y + a_{13}$$

$$y' = a_{21}x + a_{22}y + a_{23}$$

only 4 multiplications and 4 additions

- **Similarity transformations**

- Involve rotation, translation, scaling

- **Rigid-body transformations**

- Involve only translations and rotations

- Preserve <u>angles</u> and <u>lengths</u>

- General form of a rigid-body transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Properties

  * The upper 2x2 submatrix is ortonormal

    $u_1 = (r_{11}, r_{12}),\ u_2 = (r_{21}, r_{22})$

    $u_1 . u_1 = ||u_1||^2 = r_{11}{}^2 + r_{12}{}^2 = 1$

    $u_2 . u_2 = ||u_2||^2 = r_{21}{}^2 + r_{22}{}^2 = 1$

    $u_1 . u_2 = r_{11} r_{21} + r_{12} r_{22} = 0$

- Example:

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_1 . u_1 = cos(\theta)^2 + sin(-\theta)^2 = 1$$

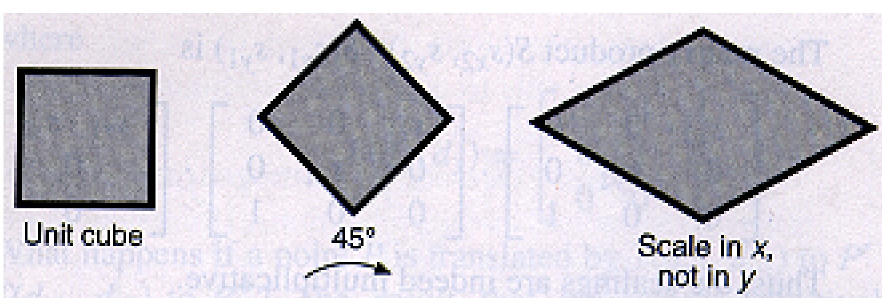
$$u_2 . u_2 = cos(\theta)^2 + sin(\theta)^2 = 1$$

$$u_1 . u_2 = cos(\theta) sin(\theta) - sin(\theta) cos(\theta) = 0$$

- **Affine transformations**

  - Involve translations, rotations, scale, and shear

  - Preserve parallelism of lines but not lengths and angles



- **Shear transformations**

- Changes the shape of the object.

- Shear along the x-axis:

$$x' = x + ay, \; y' = y$$

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Shear along the y-axis:

$$x' = x, \; y' = bx + y$$

$$SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$