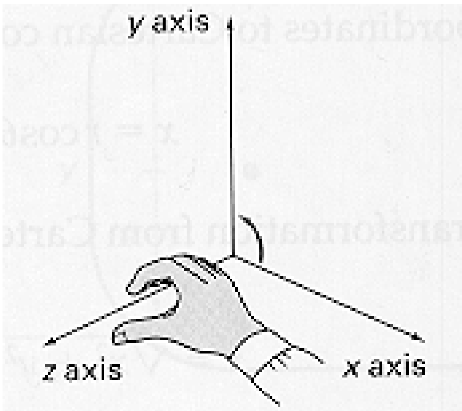
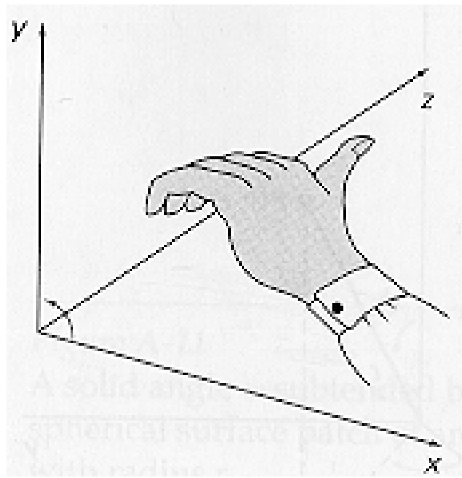
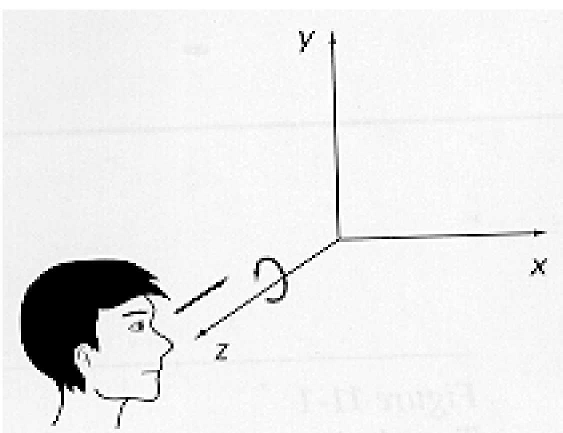
# Review of Vectors

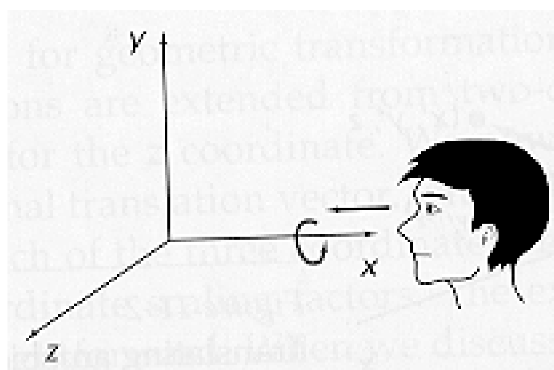
- **Right-handed** versus **left-handed** systems

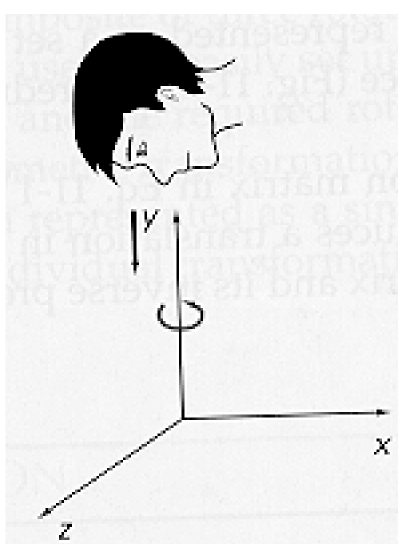



- **Positive rotation angles for right-handed systems**



- **Homogeneous coordinates of a 3D point**

  - Idea: add a third coordinate: $(x, y, z)$ --> $(x_h, y_h, z_h, w)$

  - Homogenize $(x_h, y_h, z_h, w)$:

$$x = \frac{x_h}{w}, \quad y = \frac{y_h}{w}, \; z = \frac{z_h}{w}, \; w \neq 0$$

  - In general: $(x, y, z)$ --> $(xw, yw, zw, w)$ (i.e., $x_h$=$xw$, $y_h$=$yw$, $z_h$=$zw$)

  - $w$ can assume any value ($w \neq 0$), for example, $w = 1$:

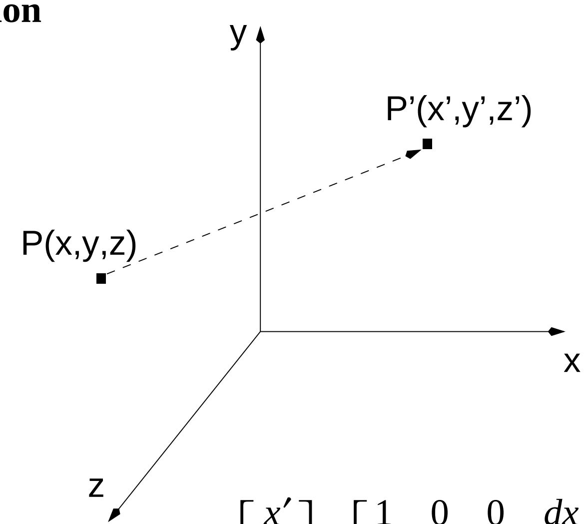
$(x, y, z) --> (x, y, z, 1)$ (no division is required when you homogenize !!)

$(x, y, z) --> (2x, 2y, 2z, 2)$ (division is required when you homogenize !!)

  - Each point $(x, y, z)$ correponds to a line in the 4D-space of homogeneous coordinates

# 3D Geometrical Transformations

- **Translation**




$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(Verification: $x' = 1x + 0y + 0z + 1dx = x + dx$)

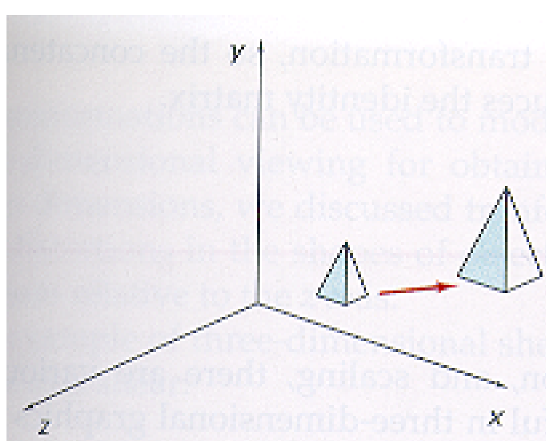
(Verification: $y' = 0x + 1y + 0z + 1dy = y + dy$)

(Verification: $z' = 0x + 0y + 1z + 1dz = z + dz$)

(Homogenize: divide by 1 !!)

$\underline{P' = T(dx, dy, dz)\ P}$

- **Scale**

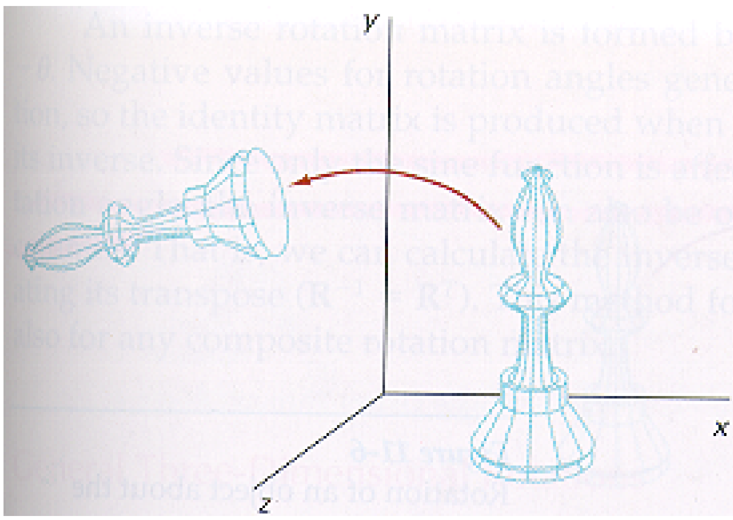


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = S(s_x, s_y, s_z)\ P}$$

- **Rotation**

- Rotation about the z-axis




$$x' = x\cos(\theta) - y\sin(\theta)$$

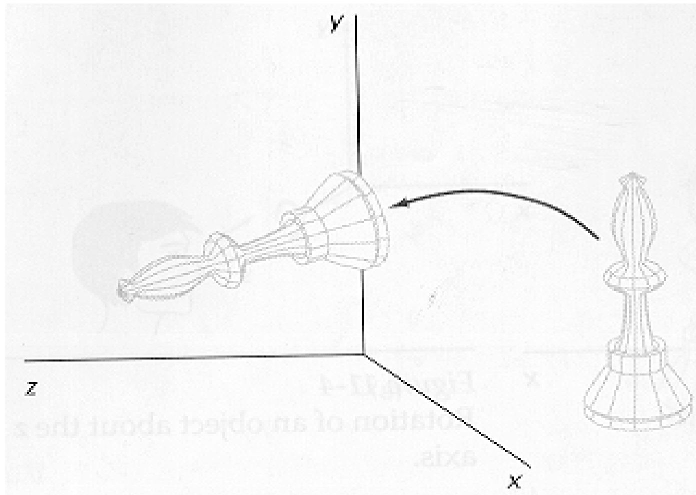
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_z(\theta)\ P}$$

- Rotation about the x-axis




$$x' = x$$

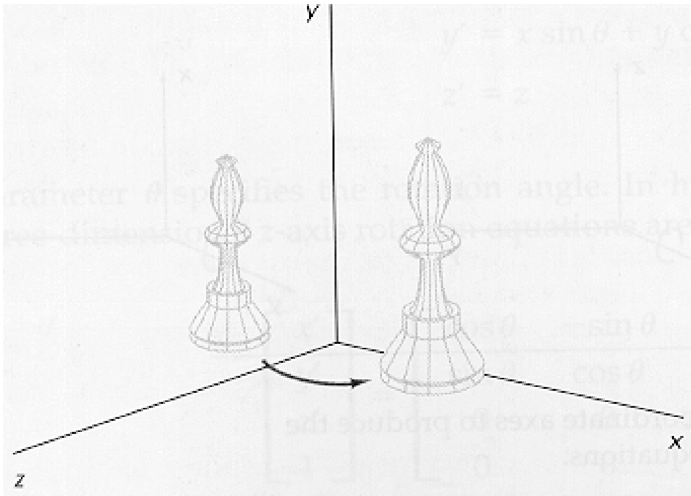
$$y' = y\cos(\theta) - z\sin(\theta)$$

$$z' = y\sin(\theta) + z\cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_x(\theta)\ P}$$

- <u>Rotation about the y-axis</u>




$$x' = z\sin(\theta) + x\cos(\theta)$$

$$y' = y$$

$$z' = z\cos(\theta) - x\sin(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

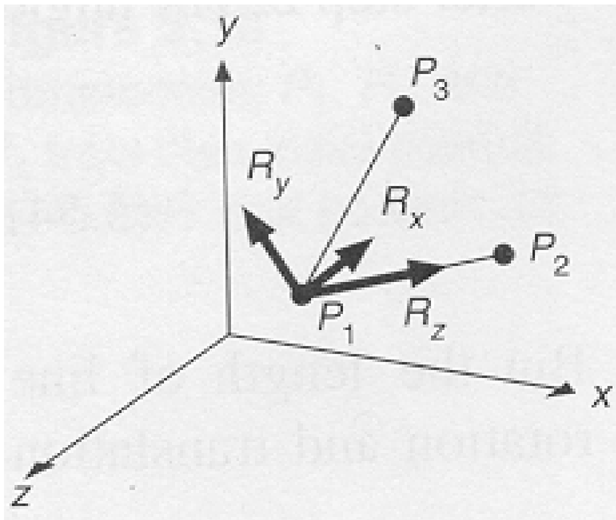
$$\underline{P' = R_y(\theta)\ P}$$

- **Change of coordinate systems**

- Suppose that you know the coordinates of $P_3$ in the $xyz$ system and that you need its coordinates in the $R_x R_y R_z$ system.

* You need to recover the transformation $T$ from $R_x R_y R_z$ to $xyz$.

* Apply $T$ on $P_3$ to compute its coordinates in the $R_x R_y R_z$ system.




- Assume that $u_x$, $u_y$, and $u_z$ are the unit vectors in the $xyz$ coordinate system.

- Assume that $r_x$, $r_y$, and $r_z$ are the unit vectors in the $R_x R_y R_z$ coordinate system (**important:** $r_x$, $r_y$, and $r_z$ are represented in the $xyz$ coordinate system).

- Find a mapping that will map $r_z \rightarrow u_z$, $r_y \rightarrow u_x$, and $r_y \rightarrow u_y$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{z_x} \\ r_{z_y} \\ r_{z_z} \\ 1 \end{bmatrix} \quad (r_z \rightarrow u_z) \qquad (1)$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{x_x} \\ r_{x_y} \\ r_{x_z} \\ 1 \end{bmatrix} \quad (r_x \rightarrow u_x) \qquad (2)$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{y_x} \\ r_{y_y} \\ r_{y_z} \\ 1 \end{bmatrix} \quad (r_y \to u_y) \qquad (3)$$

From (1): $a_3^T r_z = 1$ or $a_3 = r_z$

From (2): $a_1^T r_x = 1$ or $a_1 = r_x$

From (3): $a_2^T r_y = 1$ or $a_2 = r_y$

$$\text{Thus, } R = \begin{bmatrix} & r_x^T & & 0 \\ & r_y^T & & 0 \\ & r_z^T & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Verify

$$R \begin{bmatrix} r_x \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R \begin{bmatrix} r_y \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$R \begin{bmatrix} r_z \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$