The Geometry of Perspective Projection

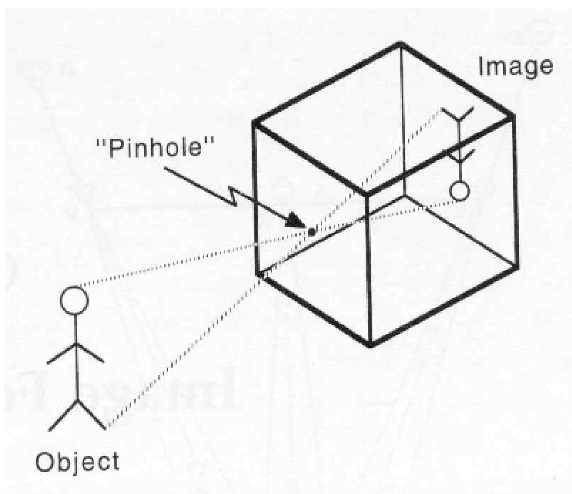
• Pinhole camera and perspective projection

- This is the simplest imaging device which, however, captures accurately the geometry of perspective projection.

- Rays of light enters the camera through an infinitesimally small aperture.

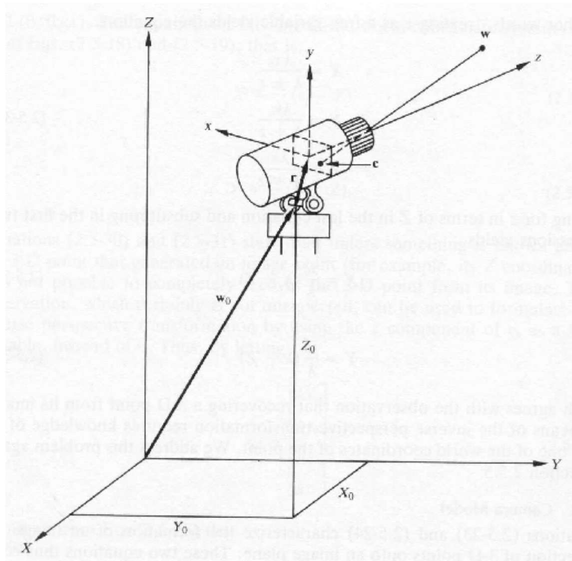
- The intersection of the light rays with the image plane form the image of the object.

- Such a mapping from three dimensions onto two dimensions is called *perspective projection*.



• A simplified geometric arrangement

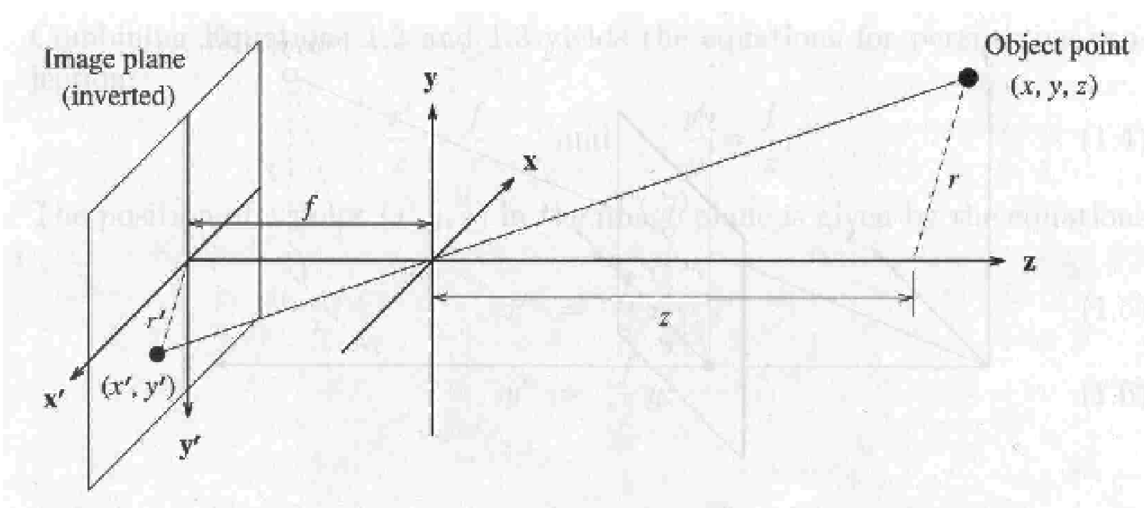
- In general, the world and camera coordinate systems are not aligned.



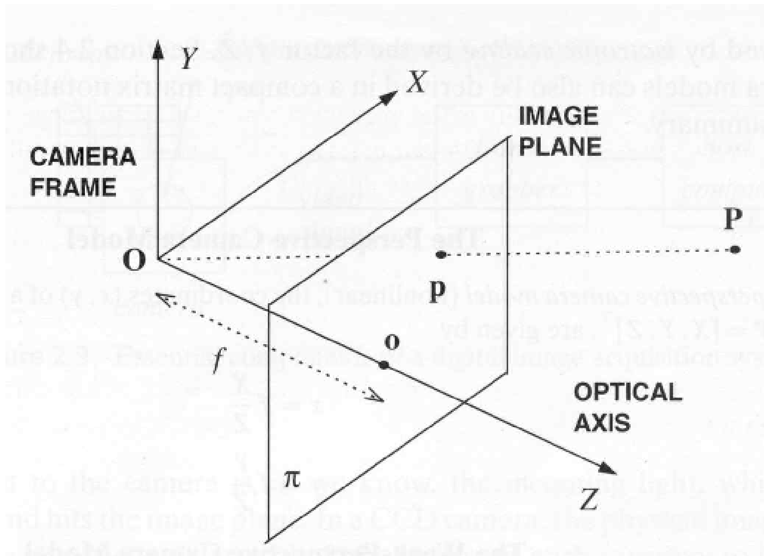
- To simplify the derivation of the perspective projection equations, we will make the following assumptions:

(1) the center of projection coincides with the origin of the world.

(2) the camera axis (optical axis) is aligned with the world's z-axis.



(3) avoid image inversion by assuming that the image plane is in front of the center of projection.



• Some terminology

- The model consists of a plane (image plane) and a 3D point O (*center of projection*).

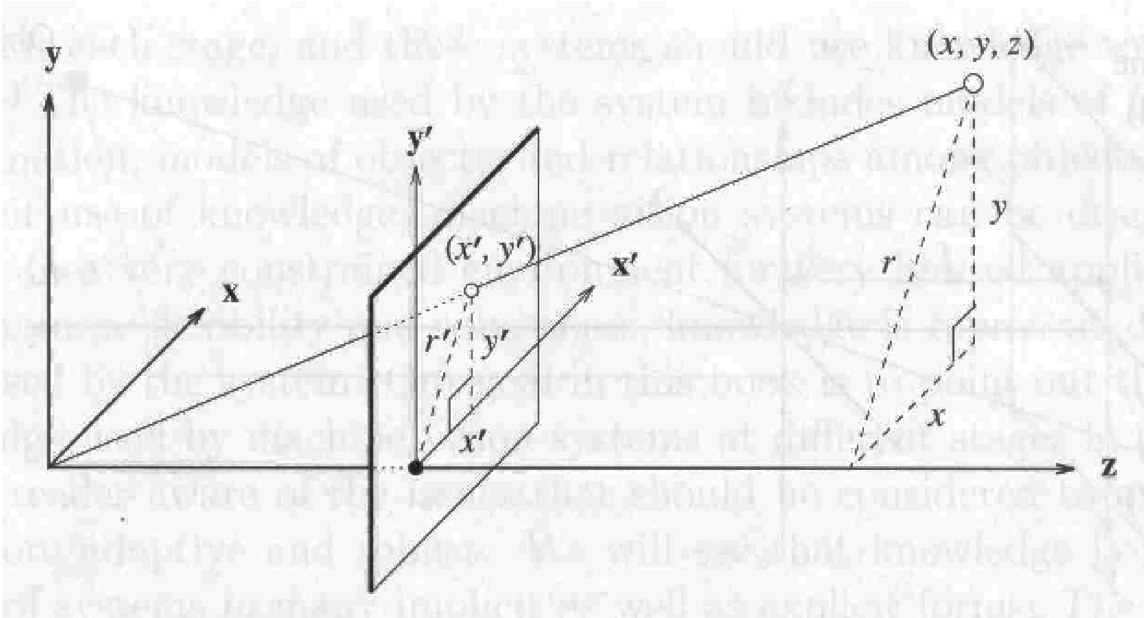
- The distance f between the image plane and the center of projection O is the *focal length* (e.g., the distance between the lens and the CCD array).

- The line through O and perpendicular to the image plane is the *optical axis*.

- The intersection of the optical axis with the image place is called *principal point* or *image center*.

(note: the principal point is not always the "actual" center of the image)

• The equations of perspective projection



(*notation*: $(x, y, z) \rightarrow (X, Y, Z)$, $r \rightarrow R$, $(x', y', z') \rightarrow (x, y, z)$, $r' \rightarrow r$)

- Using the following similar triangles:

(1) from $OA'B'$ and OAB: $\frac{f}{Z} = \frac{r}{R}$

(2) from $A'B'C'$ and ABC: $\frac{x}{X} = \frac{y}{Y} = \frac{r}{R}$

perspective proj. eqs: $x = \frac{Xf}{Z}$ $\quad y = \frac{Yf}{Z}$ $\quad z = f$

- Using matrix notation:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

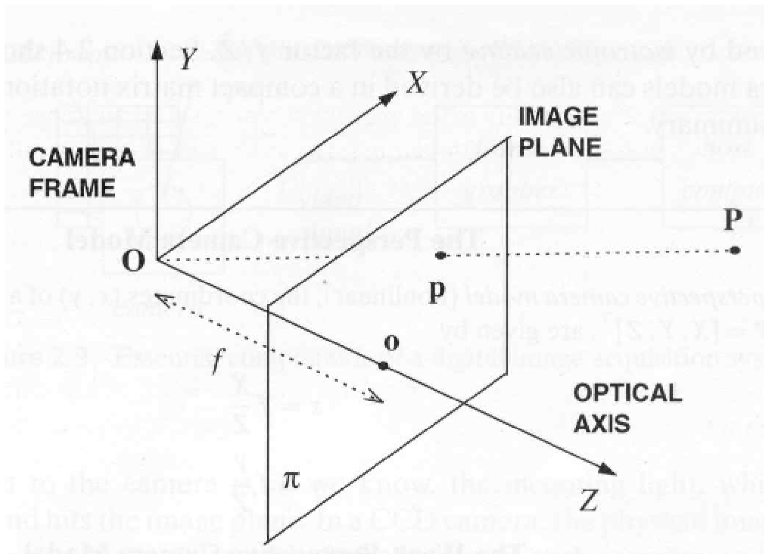
- Verify the correctness of the above matrix (homogenize using $w = Z$):

$$x = \frac{x_h}{w} = \frac{fX}{Z} \qquad y = \frac{y_h}{w} = \frac{fY}{Z} \qquad z = \frac{z_h}{w} = f$$

• Properties of perspective projection

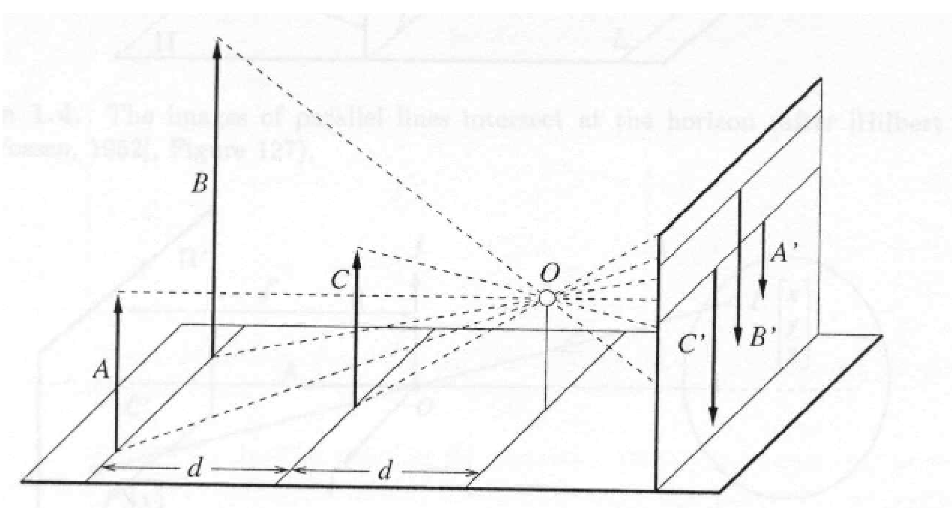
Many-to-one mapping

- The projection of a point is *not* unique (any point on the line *OP* has the same projection).



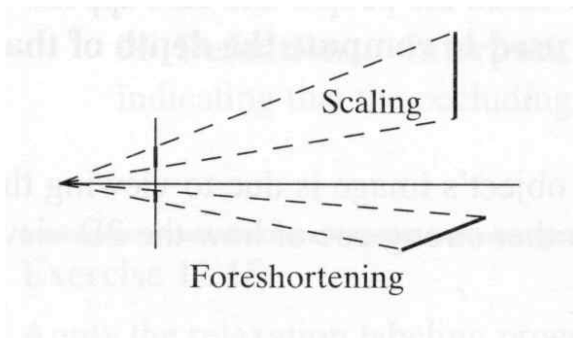
Scaling/Foreshortening

- The distance to an object is inversely proportional to its image size.



- When a line (or surface) is parallel to the image plane, the effect of perspective projection is *scaling*.

- When an line (or surface) is not parallel to the image plane, we use the term *foreshortening* to describe the projective distortion (i.e., the dimension parallel to the optical axis is compressed relative to the frontal dimension).



Effect of focal length

- As f gets smaller, more points project onto the image plane (*wide-angle camera*).

- As f gets larger, the field of view becomes smaller (more *telescopic*).

Lines, distances, angles

- Lines in 3D project to lines in 2D.

- Distances and angles are *not* preserved.

- Parallel lines *do not* in general project to parallel lines (unless they are parallel to the image plane).

Vanishing point

* parallel lines in space project perspectively onto lines that on extension intersect at a single point in the image plane called *vanishing point* or *point at infinity*.

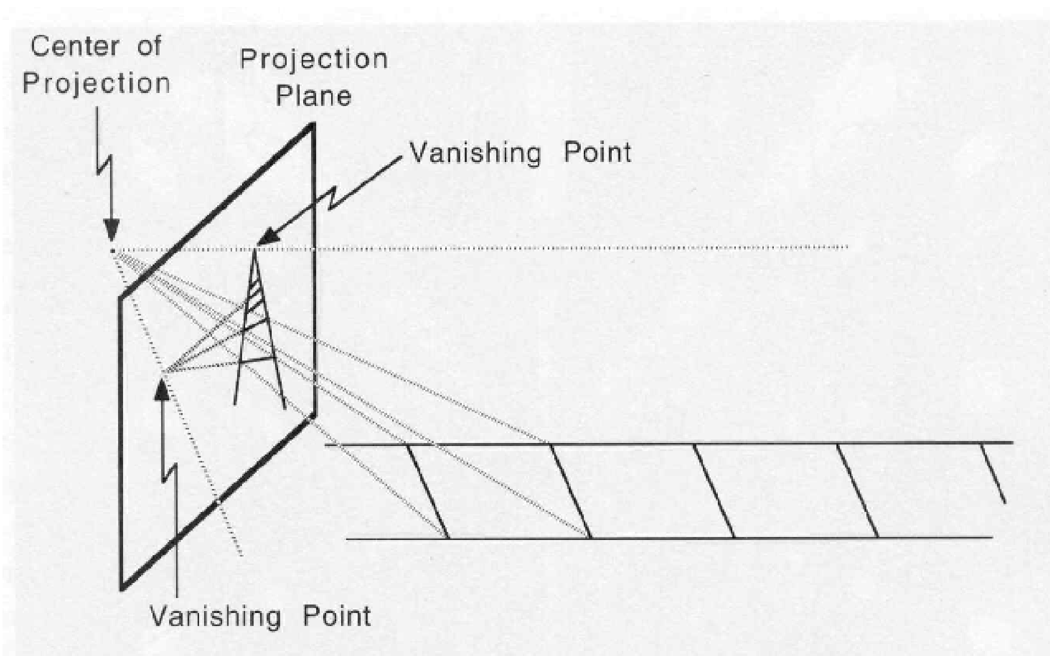
* (alternative definition) the vanishing point of a line depends on the orientation of the line and not on the position of the line.

* the vanishing point of any given line in space is located at the point in the image where a parallel line through the center of projection intersects the image plane.

Vanishing line

* the vanishing points of all the lines that lie on the same plane form the *vanishing line.*

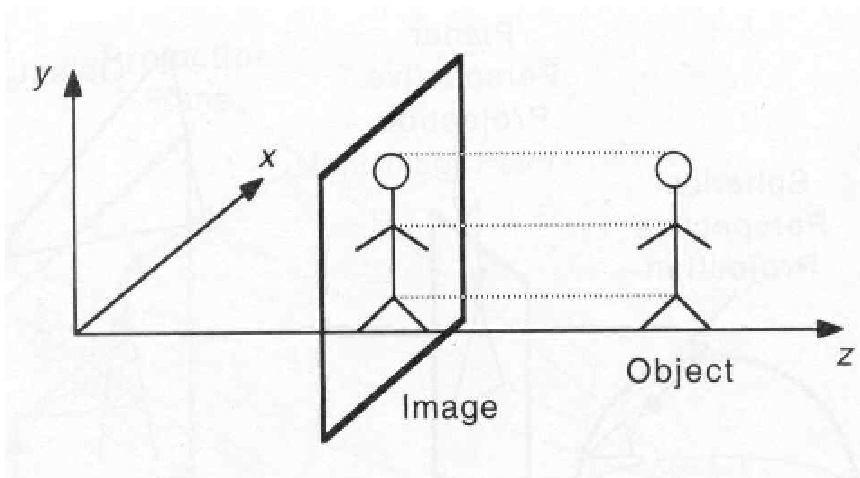
* also defined by the intersection of a parallel plane through the center of projection with the image plane.



Orthographic Projection

- It is the projection of a 3D object onto a plane by a set of parallel rays orthogonal to the image plane.

- It is the limit of perspective projection as $f -> \infty$ (i.e., $f/Z -> 1$)



orthographic proj. eqs: $x = X, \quad y = Y$ (drop Z)

- Using matrix notation:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Verify the correctness of the above matrix (homogenize using w=1):

$$x = \frac{x_h}{w} = X \qquad y = \frac{y_h}{w} = Y$$

• Properties of orthographic projection

- Parallel lines project to parallel lines.

- Size does not change with distance from the camera.

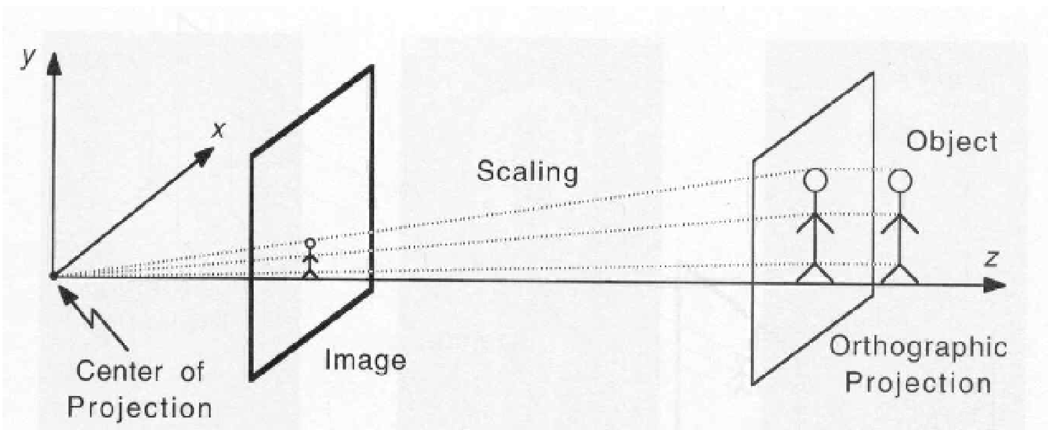
Weak Perspective Projection

- Perspective projection is a non-linear transformation.

- We can approximate perspective by scaled orthographic projection (i.e., linear transformation) if:

  (1) the object lies close to the optical axis.

  (2) the object's dimensions are small compared to its average distance $\bar{Z}$ from the camera (i.e., $\delta z < \bar{Z}/20$)



weak perspective proj. eqs: $x = \frac{Xf}{Z} \approx \frac{Xf}{\bar{Z}}$ $\quad y = \frac{Yf}{Z} \approx \frac{Yf}{\bar{Z}}$ (drop Z)

- The term $\frac{f}{\bar{Z}}$ is a scale factor now (e.g., every point is scaled by the same factor).

- Using matrix notation:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{Z} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Verify the correctness of the above matrix (homogenize using $w = \bar{Z}$):

$$x = \frac{x_h}{w} = \frac{fX}{\bar{Z}} \quad y = \frac{y_h}{w} = \frac{fY}{\bar{Z}}$$