The reconstruction problem

Both intrinsic and extrinsic parameters are known: we can solve the reconstruction problem unambiguously by triangulation.

Only the intrinsic parameters are known: we can solve the reconstruction problem but only up to an unknown scaling factor.

Neither the extrinsic nor the intrinsic parameters are available: we can solve the reconstruction problem but only up to an unknown, global projective transformation.

Reconstruction by triangulation

Assumptions and problem statement

<table>
<thead>
<tr>
<th>Both the extrinsic and intrinsic camera parameters are known.</th>
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<tr>
<td>Compute the location of the 3D points from their projections $p_l$ and $p_r$.</td>
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• What is the solution?

- The point $P$ lies at the intersection of the two rays from $O_l$ through $p_l$ and from $O_r$ through $p_r$. 

![Diagram showing the intersection of epipolar lines and epipolar planes]
**Practical difficulties**

- The two rays will not intersect exactly in space because of errors in the location of the corresponding features.

- Find the point that is closest to both rays.

- This is the midpoint $P'$ of line segment being perpendicular to both rays.

**Parametric line representation (review)**

- The parametric representation of the line passing through $P_1$ and $P_2$ is given by:

$$P(t) = P_1 + t(P_2 - P_1) \quad \text{or}$$

$$x(t) = x_1 + t(x_2 - x_1) \quad \text{and} \quad y(t) = y_1 + t(y_2 - y_1)$$

- The direction of the line is given by the vector $t(P_2 - P_1)$
• Computing $P'$

- Parametric representation of the line passing through $O_l$ and $p_l$:

$$O_l + a(p_l - O_l) = ap_l$$

- Parametric representation of the line passing through $O_r$ and $p_r$:

$$O_r^L + b(p_r^L - O_r^L) = T + bR^T p_r$$

since $O_r^L = R^T O_r^R + T = T$ and $p_r^L = R^T p_r^R + T$

$$(p_r^R = p_r, \quad p_l^L = p_l)$$

- Suppose the endpoints $P_1$ and $P_2$ are given by

$$P_1 = a_0p_l \quad \text{and} \quad P_2 = T + b_0 R^T p_r$$

- The parametric equation of the line passing through $P_1$ and $P_2$ is given by:

$$P_1 + c(P_2 - P_1)$$

- The desired point $P'$ (midpoint) is computed for $c = 1/2$
• **Computing** \( a_0 \) and \( b_0 \)

- Consider the vector \( w \) orthogonal to both \( l \) and \( r \) is given by:

\[
w = (p_l - O_l) \times (p_r^L - O_r^L) = p_l \times R^T p_r
\]

(since \( p_r^R = R(p_r^L - O_r^L) \))

- The line \( s \) going through \( P_1 \) with direction \( w \) is given by:

\[
a_0 p_l + cw = a_0 p_l + c(p_l \times R^T p_r)
\]

- The lines \( s \) and \( r \) intersect at \( P_2 \):

\[
a_0 p_l + c_0(p_l \times R^T p_r) = T + b_0 R^T p_r
\]

(assume \( s \) passes through \( P_2 \) for \( c = c_0 \))

- We can obtain \( a_0 \) and \( b_0 \) by solving the following system of equations:

\[
a_0 p_l - b_0 R^T p_r + c_0(p_l \times R^T p_r) = T
\]
Reconstruction up to a scale factor

Assumptions and problem statement

Only the intrinsic camera parameters are known. We have established \( n \geq 8 \) correspondences to compute \( E \). Compute the location of the 3D points from their projections \( p_l \) and \( p_r \).

Comments:

We cannot recover the true scale of the viewed scene since we do not know the baselile \( T \) (recall: \( Z = f \frac{T}{d} \)).

Reconstruction is unique only up to an unknown scaling factor.

This factor can be determined if we know the distance between two points in the scene.

Estimate \( E \)

- We can estimate \( E \) using the 8-point algorithm.
- The solution is unique up to an unknown scale factor.

Recover \( T \)

\[
E^T E = S^T R^T RS = S^T S = \begin{bmatrix}
T_y^2 + T_z^2 & -T_x T_y & -T_x T_z \\
-T_y T_x & T_z^2 + T_x^2 & -T_y T_z \\
-T_z T_x & -T_z T_y & T_x^2 + T_y^2
\end{bmatrix}
\]

note: \( \text{Tr}(E^T E) = 2\|T\|^2 \) or \( \|T\| = \sqrt{\text{Tr}(E^T E)/2} \)

To simplify the recovery of \( T \), let’s divide \( E \) by \( \|T\| \):

\[
\hat{E}^T \hat{E} = \begin{bmatrix}
1 - \hat{T}_x^2 & -\hat{T}_x \hat{T}_y & -\hat{T}_x \hat{T}_z \\
-\hat{T}_y \hat{T}_x & 1 - \hat{T}_y^2 & -\hat{T}_y \hat{T}_z \\
-\hat{T}_z \hat{T}_x & -\hat{T}_z \hat{T}_y & 1 - \hat{T}_z^2
\end{bmatrix}
\]

where \( \hat{T} = T/\|T| \)

(we can compute \( \hat{T} \) much easier from \( \hat{E}^T \hat{E} \))
Recover $R$

It can be shown that

$$R_i = w_i + w_j x w_k \text{ where } w_i = \hat{E}_i x \hat{T}$$

Ambiguity in $(\hat{T}, R)$

There is a twofold ambiguity in the sign of $\hat{E}$ and $\hat{T}$

There will be four different estimates for $(\hat{T}, R)$

3D reconstruction

Let’s compute the $z$ coordinate of each point in the left camera frame:

$$p_l = f_l \frac{P_l}{Z_l} \quad \text{and} \quad p_r = f_r \frac{P_r}{Z_r} = f_r \frac{R(P_l - \hat{T})}{R_3^T(P_l - \hat{T})} \quad \text{(since } P_r = R(P_l - T))$$

The first component $x_r$ of $p_r$ is:

$$x_r = f_r \frac{R_3^T(P_l - \hat{T})}{R_3^T(P_l - \hat{T})}$$

Substituting $P_l = \frac{p_l Z_l}{f_l}$ into the above equation we get:

$$Z_l = f_l \frac{(f_r R_1 - x_r R_3)^T \hat{T}}{(f_r R_1 - x_r R_3)^T p_l}$$

Compute $X_l$ and $Y_l$ from $P_l = \frac{p_l Z_l}{f_l}$

Compute $(X_r, Y_r, Z_r)$ from $P_r = R(P_l - T)$
Algorithm

Input: a set of corresponding points $p_l$ and $p_r$

1. Estimate $E$

2. Recover $\Hat{T}$

3. Recover $R$

4. Reconstruct $Z_l$ and $Z_r$

5. If the signs of $Z_l$ and $Z_r$ are:
   - 5.1 both negative for some point, change the sign of $\Hat{T}$ and goto step 4
   - 5.2 one negative, one positive, for some point, change the sign of each entry of $\hat{E}$ and goto step 3.
   - 5.3 both positive for all points, exit.

Comment: the algorithm should not go through more than 4 iterations (why?)
Reconstruction up to a projective transformation

- We will not discuss this case (more complicated)

- Look in the book if interested (pp. 166-170)