**Review on Eigenvalues and Eigenvectors**

- **Reading Assignments**


- **Other Books**


Review on Eigenvalues and Eigenvectors

• Definition

- The vector \( v \) is an eigenvector of matrix \( A \) and the number \( \lambda \) is an eigenvalue of \( A \) if:
  \[ Av = \lambda v \]  
  (assuming the non-trivial solution \( v = 0 \))

- The linear transformation implied by \( A \) cannot change the direction of the eigenvectors, only their magnitude.

• Characteristic polynomial

- To find the eigenvalues \( \lambda \) of a matrix \( A \), find the roots of the characteristic polynomial:
  \[ \det(A - \lambda I) = 0 \]

Example: \( A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix} \)

\[ \det\left( \begin{bmatrix} 5 - \lambda & -2 \\ 6 & -2 - \lambda \end{bmatrix} \right) = 0 \] or \( \lambda^2 - 3\lambda + 2 = 0 \) or \( \lambda_1 = 1, \lambda_2 = 2 \)

\[ v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \]
• Some properties

- Eigenvalues and eigenvectors are only defined for square matrices ($m = n$)
- The eigenvectors are not unique (e.g., if $v$ is an eigenvector, so is $kv$)
- Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$, then:
  
  (1) $\sum_i \lambda_i = \text{tr}(A)$

  (2) $\prod_i \lambda_i = \text{det}(A)$

  (3) if $\lambda = 0$ is an eigenvalue, then the matrix is not invertible

  (4) $A$ and $A^2$ have the same eigenvectors

  (5) if $\lambda$ is an eigenvalue of $A$, then $\lambda^2$ is an eigenvalue of $A^2$

  (6) a matrix $A$ with positive eigenvalues is called positive definite (the following is true: $x^T Ax > 0$ for every $x \neq 0$)

• Diagonalization

- The problem is finding an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix (i.e., $P$ diagonalizes $A$)

- Consider the matrix $P = [v_1 \ v_2 \ \cdots \ v_n]$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ (assume they are distinct) and $v_1 \ v_2 \ \cdots \ v_n$ are the eigenvectors of $A$:

  $AP = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix}$ or $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix}$

  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
\[
\lambda_1 = 0, \lambda_2 = 2, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}
\]
\[
P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}
\]

- **Are all \( nxn \) matrices diagonalizable?**

  - \( A \) is diagonalizable if it has \( n \) linearly independent eigenvectors (these vectors form a basis too!)

  - If \( A \) has \( n \) distinct eigenvalues, then the corresponding eigenvectors are linearly independent.

  - In general, the multiplicity of an eigenvalue should be equal to the number of eigenvectors corresponding to this eigenvalue.

- **Decomposition**

  - Let us assume that \( A \) is diagonalizable, it’s easy to see that:

  \[
  A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_n \end{bmatrix} P^{-1}
  \]
• The case of symmetric matrices

- The eigenvalues of symmetric matrices are all real, though not necessarily positive.

- The eigenvectors corresponding to distinct eigenvalues are orthogonal.

- Any symmetric $n \times n$ matrix $A$ can be written as:

$$A = V D V^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

- $V$ is an orthonormal matrix whose columns are the "normalized" eigenvectors of $A$ (i.e., using Gram-Schmidt normalization) and $D$ is a diagonal matrix containing the eigenvalues $\lambda_i$ of $A$.

• Vector representation in the eigenvector space of $A$

- The following linear transformation represents a vector in the space of eigenvectors of $A$:

$$y_i = V_i^T x$$

• Whitening transformation

$$y_i = V D^{-1/2} x_i$$

- All the eigenvalues after a whitening transformation become identical.

• The case of non-square matrices

- We can extend the results of matrix diagonalization/decomposition to the case of non-square matrices using Singular Value Decomposition (SVD).