Wavelet Transform

• Reading Assignments


• Case Studies

Christopher M. Brislawn, The FBI Fingerprint Image Compression Standard (see webpage)


Wavelet Transform

• Limitations of Fourier analysis

  - Fourier analysis cannot not provide simultaneous time and frequency localization.

  * Frequency domain localization: if the signal is a sinusoidal, it is better localized in the Fourier domain.

  * Small changes in frequency will produce changes everywhere in the spatial domain.

  (impossible to say where in time the high frequencies occur)

  * Spatial domain localization: if the signal is a simple square pulse, then it is better localized in the spatial domain.

  * Small changes in the spatial domain will produce changes everywhere in the frequency domain.
- In Fourier analysis, the basis functions are set (i.e., we cannot "customize" them for a particular application).

- Fourier analysis is more appropriate for periodic signals or signals whose statistical characteristics do not change over time; most signals found in practice, however, are finite, aperiodic, and time-varying.

- It takes a large number of Fourier components to represent a discontinuity or a sharp corner.

**• Multiresolution methods and wavelets**

- Many signals or images contain features at various scales, it is difficult to analyze them without some kind of multiresolution analysis.

![hello](a) ![hello](b) ![handwritten](c)

- A wavelet representation of a function consists of a coarse overall approximation together with detail coefficients that influence the function at various scales.

- This allows for a more accurate *local description* and separation of signal characteristics.
• Wavelet expansion

- A wavelet is a "small wave", which has its energy concentrated in time (this is in contrast to sinusoidals which have infinite energy).

- Any given decomposition of a signal into wavelets involves just a pair of waveforms (mother wavelets).

- The two shapes are translated and scaled to produce wavelets (wavelet basis) at different locations (positions) and on different scales (durations).

- The wavelet transform represents a signal as a sum of wavelets with different locations and scales.

\[
f(t) = \sum_i \sum_j c_i^j \phi(2^j t - i) + \sum_i \sum_j d_i^j \psi(2^j t - i) =
\]

\[
\sum_i \sum_j c_i^j \phi_i^j(t) + \sum_i \sum_j d_i^j \psi_i^j(t)
\]

where time or space is parameterized by \(i\) and scale by \(j\).

*Note:* for the Fourier series, there are only two possible values of \(i\), zero and \(\pi/2\); the values \(j\) give the frequency harmonics.
- Multiresolution properties of wavelet expansion

- Fine-scale wavelets are narrow and brief and coarse-scale wavelets, wide and long-lasting.

(1) The location of the wavelet moves as \( i \) changes (this allows the expansion to explicitly represent the location of events in time)

(2) The shape of the wavelet changes in scale as \( j \) changes (this allows a representation of detail or resolution).

(3) As the scale becomes finer, the steps in time become smaller (this allows representation of greater detail of higher resolution).

- If the basis functions are made half as wide and translated in steps half as wide, they will represent a larger class of signals exactly or give a better
approximation of any signal.
• Properties of wavelets

Time-frequency localization: wavelets allow simultaneous time and frequency analysis (i.e., it can tell us what the location of a feature is in the time domain as well as what its frequencies are).

Multiresolution conditions: if a set of functions can be represented by a weighted sum of $\psi(x - i)$, then a larger set (including the original) can be represented by a weighted sum of $\psi(2x - i)$.

Sparsity: for functions typically found in practice, many of the coefficients in a wavelet representation are either zero or very small.

Linear-time complexity: transforming to and from a wavelet representation can generally be accomplished in linear time.
Adaptability: unlike Fourier techniques, wavelets can be adapted to represent a wide variety of functions (e.g., functions with discontinuities, functions defined on bounded domains etc.).

(1) Wavelets are well suited to problems involving images, open or closed curves, and surfaces of just about any variety.

(2) In contrast to Fourier analysis, wavelets can represent functions with discontinuities or corners more efficiently (i.e., some have sharp corners themselves).

• A simple example (Haar basis)

- Suppose we are given a 1D "image" with a resolution of 4 pixels:

  \[ [9 \ 7 \ 3 \ 5] \]

- This image can be represented in the Haar basis as follows:

  1. Start by averaging the pixels together (pairwise) to get a new lower resolution image:

      \[ [8 \ 4] \] (averaged and subsampled)

- To be able to recover the original four pixels from the two averaged pixels, we need to store some detail coefficients, which capture the missing information.

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Averages</th>
<th>Detail Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>[9 \ 7 \ 3 \ 5]</td>
<td>[]</td>
</tr>
<tr>
<td>2</td>
<td>[8 \ 4]</td>
<td>[1 \ -1]</td>
</tr>
</tbody>
</table>
2. Repeating this process on the averages gives the full decomposition:

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<td>[8 4]</td>
<td>[1 - 1]</td>
</tr>
<tr>
<td>4</td>
<td>[6]</td>
<td>[2]</td>
</tr>
</tbody>
</table>

3. The Harr decomposition of the original four-pixel image is:

\[[6 2 1 - 1]\]

- We can reconstruct the original image to any resolution by adding or subtracting the detail coefficients from the lower-resolution versions.
• 1D Haar wavelets (formal treatment)

- Let’s think of 1D images as functions:

* Consider a one-pixel image to be a function that is constant over [0,1) (we will denote by $V^0$ the space of all such functions).

* Consider a two-pixel image as a function having two constant pieces over the intervals [0, 1/2) and [1/2,1) (we will denote by $V^1$ the space of all such functions).

* Continuing in this manner, $V^j$ represents all the $2^j$-pixel images (functions having constant pieces over $2^j$ equal-sized intervals on [0,1))

* $V^0, V^1, ..., V^j$ define a hierarchical representation.

\[
V^0 \subset V^1 \subset V^2 \cdots
\]

(every vector in $V^j$ is also contained in $V^{j+1}$)
Let's define a basis for $V^j$:

$$\phi^j_i(x) := \phi(2^j x - i), \quad i = 0, 1, \ldots, 2^j - 1$$

(scaled and translated versions of the box function below)

$$\phi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}$$

Suppose $W^j$ is the orthogonal complement of $V^j$ in $V^{j+1}$ (i.e., all the functions in $V^{j+1}$ that are orthogonal to all the functions in $V^j$)
- Let's define a basis $\psi^j_i$ for $W^j$:

$$
\psi^j_i(x) := \psi(2^j x - i), \quad i = 0, 1, \ldots, 2^j - 1
$$

$$
\psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1/2 \\
-1 & \text{if } 1/2 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}
$$

wavelets for $W^1$.

- Comments about the basis functions $\psi^j_i$:

* The basis functions $\psi^j_i$ of $W^j$ and the basis functions $\phi^j_i$ of $V^j$ form a basis for $V^{j+1}$.

* We can think of the $\psi^j_i$ as a means of representing the parts of a function in $V^{j+1}$ that cannot be represented in $V^j$ (i.e., the detail coefficients are really coefficients of the $\psi^j_i$).
\[ j = 3 \quad \text{vs.} \quad j = 2 \]

\[ \nu_3 = \nu_2 \oplus \nu_2 \]

\[ \phi(4t - k) \quad \nu_2 \]

\[ \psi(4t - k) \quad \nu_2 \]
\[ \begin{align*}
  &j = 2 & j = 1 \\
  &\begin{array}{c}
    \phi(4t - k) \\
    \phi(2t - k)
  \end{array} & \begin{array}{c}
    \phi(2t - k) \\
    \psi(2t - k)
  \end{array} \\
  &\nu_2 = \nu_1 \oplus \mathcal{W}_1
\end{align*} \]

\[ \begin{align*}
  &j = 1 & j = 0 \\
  &\begin{array}{c}
    \phi(2t - k)
  \end{array} & \begin{array}{c}
    \phi(t - k)
  \end{array} \\
  &\nu_1 = \nu_0 \oplus \mathcal{W}_0
\end{align*} \]
The important features of a signal can be better described not by using $\phi_i^j$ and increasing $j$ but by using $\psi_i^j$ which span the differences between the spaces spanned by $\phi_i^j$.

- Decompositions of $f(x)$

using the basis functions in $V^2$

$$f(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x)$$
using the basis functions in $V^1$ and $W^1$

$$V^2 = V^1 + W^1$$

$$f(x) = c_0^1 \phi_0^1(x) + c_1^1 \phi_1^1(x) + d_0^1 \psi_0^1(x) + d_1^1 \psi_1^1(x)$$
= 8 ×

+ 4 ×

+ 1 ×

+ −1 ×
using the basis functions in $V^0$, $W^0$ and $W^1$

$$V^2 = V^1 + W^1 = V^0 + W^0 + W^1$$

$$f(x) = c_0^0 \phi_0^0(x) + d_0^0 \varphi_0^0(x) + d_0^1 \varphi_0^1(x) + d_1^1 \varphi_1^1(x)$$

- The four functions shown above constitute the Haar wavelet basis for $V^2$. 
• An example
• **Haar scaling functions and wavelets**

- In general, a function can be represented as follows using the Haar wavelet basis:

\[
f(t) = \sum_{i=-\infty}^{\infty} c_i \phi(t - i) + \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} d_i^j \psi(2^j t - i)
\]

- The functions \( \phi(t) \) are called *scaling functions* and the functions \( \psi(t) \) are called *wavelet functions*.

- For each increasing index \( j \), a finer resolution function is added, which adds increasing detail.

• **Orthogonality and normalization**

- The Haar basis form an orthogonal basis (not always true for other wavelet bases).

- It can become orthonormal through the following normalization:

\[
\phi_i^j(x) = \sqrt{2^j} \phi(2^j x - i)
\]

\[
\psi_i^j(x) = \sqrt{2^j} \psi(2^j x - i)
\]
- The lower resolution coefficients can be calculated from the higher resolution coefficients by a tree-structured algorithm called a **filter bank**.

- The filter implemented by $h_0(-n)$ is a lowpass filter and the one implemented by $h_1(-n)$ is a highpass filter.

- The higher resolution coefficients can be calculated from the lower resolution coefficients too.
Figure 3.6. Two-Band Synthesis Bank
• **Lowpass - Highpass filters**

**Haar**

LOWPASS: \( \frac{1}{\sqrt{2}} [1 \ 1] \)

HIGHPASS: \( \frac{1}{\sqrt{2}} [1 \ -1] \)

**Daubechies**

LOWPASS: \( \frac{1}{4\sqrt{2}} [1 + \sqrt{3} \ 3 + \sqrt{3} \ 3 - \sqrt{3} \ 1 - \sqrt{3}] \)

HIGHPASS: \( \frac{1}{4\sqrt{2}} [1 - \sqrt{3} \ \sqrt{3} - 3 \ 3 + \sqrt{3} \ - 1 - \sqrt{3}] \)

• **2D Haar basis**

- To construct the 2D Haar wavelet basis, we need to consider all possible outer products of 1D basis functions (standard Haar basis).
• **Methodology using filter banks**

- Many filters, including the Haar basis, are *separable* which means that we can convolve their 1D counterparts first with the rows and then with the columns.

- The wavelet transform breaks an image down into four subsampled images (subsampling is done by keeping every other pixel).

1. Convolve the lowpass filter with the rows of the original image.

2. Convolve the highpass filter with the rows of the original image.
3. (lowpass - lowpass) Convolve the lowpass filter with the columns of the image from step 1. Subsample this result by taking every other value.

4. (lowpass - highpass) Convolve the highpass filter with the columns of the image from step 1. Subsample this result by taking every other value.

5. (highpass - lowpass) Convolve the lowpass filter with the columns of the image from step 2. Subsample this result by taking every other value.

6. (highpass - highpass) Convolve the highpass filter with the columns of the image from step 2. Subsample this result by taking every other value.

(images need to be extended periodically to avoid problems with the boundaries -circular convolution)
- The wavelet transform can be applied again on the lowpass-lowpass version of the image, yielding seven subimages.

- We can continue this procedure in order to get 10, 13 etc. subimages.

- This process is called multiresolution decomposition of the image.

- Note that the transform contains spatial information (i.e., the image itself is visible in the transform domain).
• **Inverse wavelet transform**

(1) Enlarge the wavelet transform data to its original size (insert zeros between each value).

(2) Convolve each of the four subimages with the corresponding inverse (lowpass and highpass) filters.

(3) Sum the results to obtain the original image.

(4) For the Haar filters, the inverse filters are identical to the forward filters.
(5) For the Daubechies filters, the inverse filters are

\[ LOWPASS_{inv}: \frac{1}{4\sqrt{2}} [3 - \sqrt{3}  \ 3 + \sqrt{3}  \ 1 + \sqrt{3}  \ 1 - \sqrt{3}] \]

\[ HIGHPASS_{inv}: \frac{1}{4\sqrt{2}} [1 - \sqrt{3}  \ -1 - \sqrt{3}  \ 3 + \sqrt{3}  \ -3 - \sqrt{3}] \]
- Wavelet packets

- Wavelet packets (and cosine packets) are intermediate between wavelets and sinusoids: they oscillate many times, but are still localized to a segment of the signal duration (i.e., they have a location and duration but also have an oscillation like sinusoids).

- Like wavelets and sinusoids, every signal can be represented uniquely as a sum of wavelet packets.

- They can outperform (i.e., sparser representations) classical wavelets and sinusoids in representing locally oscillatory signals or images (e.g., digitized music or texture).

- Wavelet packets can be computed in $O(N \log N)$ time (using generalizations of the filter bank algorithm).