QFT Bound Determination Using a Kharitonov and Algorithmic Framework

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Abstract

This paper is part of the ongoing research to reduce the computational cost of quantitative feedback theory (QFT) using ideas from Kharitonov stability theory and the theory of algorithms. By limiting the analysis to the simple case of a plant transfer function with denominator uncertainty, we gain insight into the problem of determining tracking, robust stability and actuator bounds for QFT design. A coordinate transformation allows us to exploit symmetry to reduce the computational time for the problem. This concept can be applied to more general uncertainty structures and is a main contribution of this paper. Our analysis demonstrates the importance of estimating asymptotic bounds on computational time as is standard in algorithmic analysis. We were also able to divide the compensator plane into twelve different regions with each region sharing the critical points of a value set needed in the computation of QFT bounds. We apply our algorithms to bound determination for a simple position control system.

I. Introduction

Quantitative feedback theory (QFT) is an integrated design methodology that allows the design of tracking controllers in the presence of parameter uncertainty. As originally proposed, QFT required extensive experience to obtain good results [1]. Today, computer programs have made the methodology accessible to the average control engineer [2]. However, the computational cost of QFT becomes prohibitive as the number of uncertain parameters increases. Several researchers have developed interesting algorithms to reduce the time required to calculate the bounds used in QFT design [3], [4], [5]. In [3], the use of value sets known from Kharitonov stability theory [6] allowed the authors to compute the tracking bounds for affine uncertainty in the numerator and denominator. In [4], a major simplification in QFT design is achieved by relying on sensitivity or complementary sensitivity bounds rather than tracking bounds.

Fadali and LaForge [5] introduce the notation and tools of algorithmic analysis to the computation of QFT bounds. To demonstrate the usefulness of these tools, the analysis was restricted to the simple case of a system with denominator uncertainty and a numerator known with arbitrary accuracy. The paper demonstrated that to assess the relative usefulness of newly proposed algorithms, it was essential to obtain asymptotic bounds on their computational time and to formally prove their correctness as is standard practice in algorithmic design.

An important observation used in [3] and [5] is the fact that the magnitude of the sum of a test compensation value (complex number) and a plant value set can be considered as the distance between the value set and the negative of the compensation value. The tracking bound for QFT can therefore be viewed as an upper bound on the ratio of the maximum distance to the minimum distance to the value set. Similarly, the stability robustness bound can be viewed as a lower bound on the distance to the value set. In this paper, we further examine the simple problem solved in [5] exploiting the same observation. We show that in special cases the complex plane can be divided into regions where the farthest and closest points of the value set remain unchanged. A coordinate transformation simplifies the evaluation of both the stability robustness and tracking bounds by reducing the problem to one where the bounds are symmetric about the value set. The inverse transformation yields the required bounds and the entire computation is achieved in constant time. These concepts are clearly explained in the paper and demonstrated using a simple position control system model.

While the subdivision of the complex plane and the coordinate transformation are applied here to the simple problem solved in [5], the same concepts are applicable to more complex uncertainties. Application to systems with numerator interval polynomials and to affine numerator and denominator will be presented in future papers. The results are also useful if the modified QFT design approach of [4] is adopted.

The paper is organized as follows. First we examine the tracking and stability robustness bounds for a system with denominator uncertainty. Then we show that the complex plane can be divided into twelve regions each of which has a fixed farthest and closest point in the rectangular value set. Then we introduce the coordinate transformation that simplifies the computation of the required QFT bounds. Next, we discuss the computational time [7] of the new algorithm as compared to [5] and to the standard gridding approach [8]. Finally, we discuss the application of the new concepts to simple position control system.

II. QFT Bounds

Consider a single-input-single-output system comprised of a plant and a feedback compensator. The closed-loop transfer function (CLTF) for the system is given by

$$|T(\omega)| = \frac{1}{\left|P^{-1}(j\omega, p, q) + K(j\omega)\right|} \tag{1}$$

where P is the uncertain plant transfer function, K is a compensator transfer function, and p and q are vectors of uncertain parameters for the numerator and denominator polynomials, respectively. Because we are primarily interested in exploring the development of new computational tools for QFT bounds, we focus our attention on the simplest meaningful problem for which those tools can be explored. We consider the case of a plant transfer function with denominator uncertainty, i.e.

$$P(j\omega, p, q) = \frac{N(j\omega)}{D(j\omega, q)} \tag{2}$$

The maximum CLTF magnitude is therefore given by

$$|T(\omega)|_{\max} = \frac{|N(j\omega)|}{\min\limits_{q} |D(j\omega, q) + KN(j\omega)|} \tag{3}$$

while the minimum CLTF magnitude is

$$|T(\omega)|_{\min} = \frac{|N(j\omega)|}{\max\limits_{q} |D(j\omega, q) + KN(j\omega, p)|} \tag{4}$$

The worst-case magnitude ratio is

$$\frac{|T(\omega)|_{\max}}{|T(\omega)|_{\min}} = \frac{\max\limits_{q} |D(j\omega, q) + KN(j\omega)|}{\min\limits_{q} |D(j\omega, q) + KN(j\omega)|} \tag{5}$$

Let $C = x_c + j\, y_c$ be the product of K and N and rewrite (5) as

$$\frac{|T(\omega)|_{\max}}{|T(\omega)|_{\min}} = \frac{\max\limits_{q} |D(j\omega, q) + C(j\omega)|}{\min\limits_{q} |D(j\omega, q) + C(j\omega)|} \tag{6}$$

It will be convenient to re-express C in Cartesian coordinates as (x_C, y_C) and write (6) in the form

$$\frac{|T(\omega)|_{\max}}{|T(\omega)|_{\min}} = \frac{\max\limits_{q} |D(j\omega, q) - C_m(j\omega)|}{\min\limits_{q} |D(j\omega, q) - C_m(j\omega)|} \tag{7}$$

where $C_m = -C = (-x_c, -y_c) = (x, y)$. We now observe that the magnitudes in (3)-(7) involve distances between the values of the complex number D and the complex number C_m [3]. Thus the problem of determining QFT tracking bounds is equivalent to determining the ratio of the maximum to minimum distance between the point C_m and the set D.

In addition, robust stability requires that

$$|T(\omega)|_{\max} = \frac{|N(j\omega)|}{\min\limits_{q} |D(j\omega, q) - C_m|} \leq M_p \tag{8}$$

i.e.

$$\min_{q} |D(j\omega, q) - C_m| \geq \frac{|N(j\omega)|}{M_p} \tag{9}$$

In the case of cascade compensation, (7) is unchanged, but the right hand side of (9) is multiplied by $|K|$. Similarly, the lower bound

$$|T(\omega)|_{mi} = \frac{|N(j\omega)|}{\min\limits_{q} |D(j\omega, q) - C_m|} \geq M_l \tag{10}$$

is equivalent to the condition

$$\min_{q} |D(j\omega, q) - C_m| \leq \frac{|N(j\omega)|}{M_l} \tag{11}$$

Let D be the nth order interval polynomial

$$D(s; q) = a_n(q)\, s^n + a_{n-1}(q)\, s^{n-1} + \ldots + a_0(q) \tag{12}$$

with uncertainty bounds $a_i \in \left[a_i^-, a_i^+\right]$, $i = 0, 1, \ldots, n$. Then the value set associated with the uncertain polynomial is the Kharitonov rectangle D with generating polynomials [6]

$$\begin{aligned}
D^{++}(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + \cdots \\
D^{+-}(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + \cdots \\
D^{-+}(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + \cdots \\
D^{--}(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \cdots
\end{aligned} \tag{13}$$

Refer to Figure 1; $(x, y) \in D$ if and only if

$$x_- \leq x \leq x_+ \tag{14a}$$
$$y_- \leq y \leq y_+ \tag{14b}$$

where x_+ (*resp.* x_-) denotes the real component of D^{++} (*resp.* D^{--}) and y_+ (*resp.* y_-) denotes the imaginary component of D^{++} (*resp.* D^{--}).

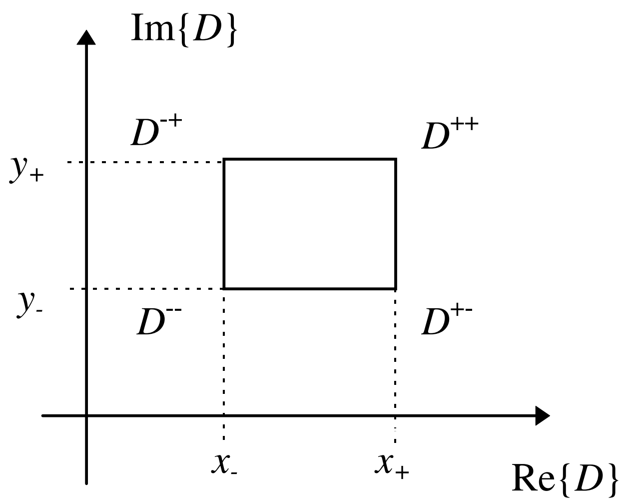

Fig. 1 Kharitonov rectangle D .

Algorithmic computation of the worst-case ratio is now tractable because, as we show, the maximum and minimum magnitude corresponds to an extreme point of the value set D , or to a point on an edge of D .

III. Determination of the Maximum and Minimum Distance

To determine the stability robustness and tracking bounds for a system of denominator uncertainty, we obtain the maximum and minimum distances between a test point $C_m = (x, y)$ in the complex plane and the Kharitonov rectangle D . Figure 2, shows a test point and the Kharitonov rectangle in the complex plane. We use the notation

$$\left(x_{av}, y_{av}\right) = \left(\frac{x_- + x_+}{2}, \frac{y_- + y_+}{2}\right) \tag{15}$$

The plane can be divided into regions 1, 2, …, 12. Within region *i,* there is a unique point $(\hat{x}, \hat{y}) \in D$ at maximum distance from C_m and a unique point $(\breve{x}, \breve{y}) \in D$ at minimum distance. Excluded from consideration is the interior of the Kharitonov rectangle since it corresponds to unstable systems. The twelve regions have the properties summarized in the Table 1 and in the following theorem.

Theorem 1. Given a rectangular region D and a test point (x, y). Then the furthest point $(\hat{x}, \hat{y})$ in D to (x, y) and the closest point $(\breve{x}, \breve{y})$ in D to (x, y) are as given in Table 1.

Proof

The distance D to (x, y) is maximized or minimized by separately maximizing

$$(x-\hat{x})^2 \quad and \qquad (y-\hat{y})^2$$

or minimizing

$$(x-\breve{x})^2 \quad and \qquad (y-\breve{y})^2$$

respectively i.e. minimizing the distance in the horizontal and vertical directions separately.

For the horizontal direction, $(x-\hat{x})^2$ is maximized in Regions 1 , 2, 5, 6, 7, 12, at $\hat{x} = x_+$, and in Regions 3, 4, 8, 9, 10, 11, at $\hat{x} = x_-$. $(x-\check{x})^2$ is minimized in Regions 1, 2, 5, 6, at $\check{x} = x_-$, and in Regions 3, 4, 9, 10, at $\check{x} = x_+$. In Regions 7, 8, 11, 12, a minimum value of zero is possible with $\check{x} = x$.

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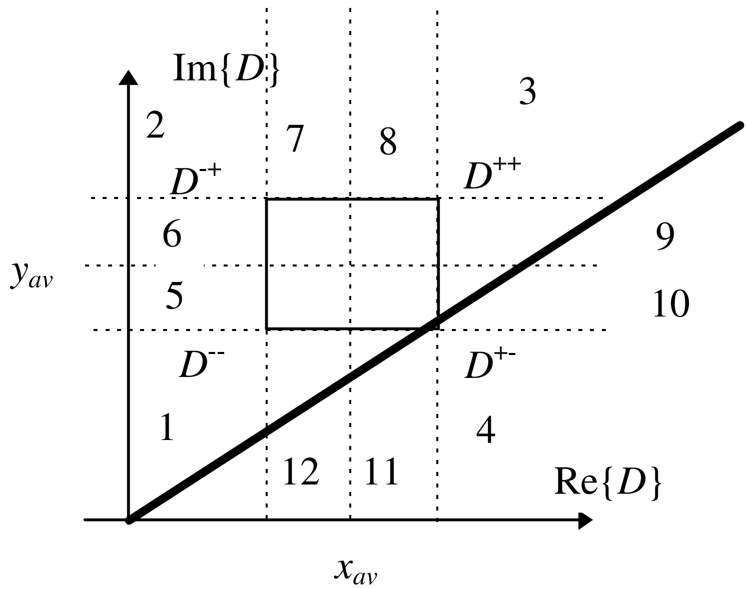

Fig. 2 Twelve regions for the determination of the maximum and minimum distances to the point (x_c, y_c).

Table 1 Properties of the regions traversed by radial lines.

Region No.	Boundary	$(\hat{x},\hat{y})$	$(\check{x},\check{y})$
1	$x < x_-$ & $y < y_-$	(x_+, y_+)	(x_-, y_-)
2	$x < x_-$ & $y > y_+$	(x_+, y_-)	(x_-, y_+)
3	$x > x_+$ & $y > y_+$	(x_-, y_-)	(x_+, y_+)
4	$x > x_+$ & $y < y_-$	(x_-, y_+)	(x_+, y_-)
5	$x < x_-$ & $y \in [y_-, y_{av}]$	(x_+, y_+)	(x_-, y)
6	$x < x_-$ & $y \in [y_{av}, y_+]$	(x_+, y_-)	(x_-, y)
7	$x \in [x_-, x_{av}]$ & $y > y_+$	(x_+, y_-)	(x, y_+)
8	$x \in [x_{av}, x_+]$ & $y > y_+$	(x_-, y_-)	(x, y_+)
9	$x > x_+$ & $y \in [y_{av}, y_+]$	(x_-, y_-)	(x_+, y)
10	$x > x_+$ & $y \in [y_-, y_{av}]$	(x_-, y_+)	(x_+, y)
11	$x \in [x_{av}, x_+]$ & $y < y_-$	(x_-, y_+)	(x, y_-)
12	$x \in [x_-, x_{av}]$ & $y < y_-$	(x_+, y_+)	(x, y_-)

For each region the tracking bound is defined by

$$R^2 = \frac{(x-\hat{x})^2 + (y-\hat{y})^2}{(x-\breve{x})^2 + (y-\breve{y})^2} \le B^2 \tag{16}$$

with $B > 1$. At the boundary in Regions 1-4 we have the equation

$$\begin{aligned} &\left(B^2-1\right)x^2 - 2x\left(B^2\breve{x}-\hat{x}\right) + \left(B^2\breve{x}^2 - \hat{x}^2\right) \\ &+\left(B^2-1\right)y^2 - 2y\left(B^2\breve{y}-\hat{y}\right) + \left(B^2\breve{y}^2 - \hat{y}^2\right) = 0 \end{aligned} \tag{17}$$

with the permissible region outside the boundary. Completing the squares gives the equation of the circle with center (c_x, c_y) and radius c_r

$$\left(x-c_x\right)^2 + \left(y-c_y\right)^2 = c_r^2 \tag{18}$$

where

$$\begin{aligned} c_x &= \frac{B^2\breve{x}-\hat{x}}{B^2-1} \\ c_y &= \frac{B^2\breve{y}-\hat{y}}{B^2-1} \\ c_r &= \frac{B}{B^2-1}\sqrt{\left(\hat{x}-\breve{x}\right)^2 + \left(\hat{y}-\breve{y}\right)^2} \end{aligned} \tag{19}$$

In Regions 5-12, one of the quadratic terms in the denominator vanishes. For Region 7,8, 11, 12, (17) becomes

$$\left(B^2-1\right)y^2 - 2y\left(B^2\breve{y}-\hat{y}\right) - \left(x-\hat{x}\right)^2 + \left(B^2\breve{y}^2 - \hat{y}^2\right) = 0 \tag{20}$$

Completing squares we have the equation of the hyperbola

$$-\frac{(x-\hat{x})^2}{\mathrm{c}_{ry}^2\left(B^2-1\right)} + \frac{\left(y-c_y\right)^2}{\mathrm{c}_{ry}^2} = 1 \tag{21}$$

where $\mathrm{c}_{ry} = B\left|\hat{y}-\breve{y}\right|/\left(B^2-1\right) = c_r$ with the x-related terms removed. The hyperbola is symmetric about axes through $\left(\hat{x}, c_y\right)$ with x-intercept at $\hat{x} \pm \sqrt{\mathrm{c}_{ry}^2\left(B^2-1\right)}$ with asymptotes through $\left(\hat{x}, c_y\right)$ and $c_y \pm \mathrm{c}_{ry}$. In Regions 5, 6, 9, 10, we have similar hyperbolas with x and y interchanged and $\mathrm{c}_{rx} = B\left|\hat{x}-\breve{x}\right|/\left(B^2-1\right) = c_r$ with the y-related terms removed.

Each boundary is only valid for the region for which the points $\left(\hat{x}, \hat{y}\right)$ and $\left(\breve{x}, \breve{y}\right)$ apply. In Regions 1-4, the outside of the circle is the allowable compensation where the ratio is less than the specified value. In the remaining Regions, the allowable compensation is outside the hyperbolas.

The stability robustness bound is defined by

$$\left(x-\breve{x}\right)^2 + \left(y-\breve{y}\right)^2 \ge \left(N/M\right)^2 \tag{22}$$

So the boundary is a circle of center $\left(\breve{x}, \breve{y}\right)$ and radius *N*/*M* and the allowable region is outside the circle.

The actuator bound is due to the maximum allowable compensation magnitude $\gamma_{\max}$ at each frequency. It is defined by the equation

$$x^2 + y^2 \le \gamma_{\max}^2 \tag{23}$$

This boundary is a circle centered at the origin with radius $\gamma_{\max}$ and the allowable region is inside the circle.

Theorem 2. Let the points $\left(\hat{x}, \hat{y}\right)$ and $\left(\breve{x}, \breve{y}\right)$ be known for a system with an interval polynomial in the denominator and a fixed numerator. Then the QFT boundaries and the permissible compensation regions are given by

1) In Region 1-4, the tracking boundary is a circle of radius c_r centered at $(-c_x, -c_y)$ with the permissible region outside the boundary. In Regions 7, 8, 11, 12, the tracking boundary is a hyperbola symmetric with respect to axes through $\left(-\hat{x}, -c_y\right)$ with horizontal intercept at $-\hat{x} \pm \sqrt{B^2 - 1}$ and asymptotes through $\left(-\hat{x}, -c_y\right)$ and $-c_y \pm c_r$. In Regions 5, 6, 9, 10, the tracking boundary is a hyperbola symmetric with respect to axes through $\left(-c_x, -\hat{y}\right)$ with horizontal intercept at $-\hat{y} \pm \sqrt{B^2 - 1}$ and asymptotes through $\left(-c_x, -\hat{y}\right)$ and $-c_x \pm c_{rx}$.

2) The stability robustness boundary is a circle of radius *N*/*M* centered at $\left(-\breve{x}, -\breve{y}\right)$ with the permissible region outside the boundary.

3) The actuator boundary is a circle of radius $\gamma_{\max}$ centered at the origin with the permissible region inside the boundary.

Proof
The results are obtained on replacing (x, y) by $(-x, -y)$ in (16), (17), (18).

■

Remark 1
Theorem 2 is only valid if the farthest and closest points in the Kharitonov rectangle to a test point are fixed. In each of Regions 5-12 this is not the case. The result is that the robust stability boundary in these regions are actually *straight lines* parallel to the adjacent edge of Kharitonov rectangle and tangent to the set of circles corresponding to the region. This conclusion and others are much more easily reached by transforming the problem as described in the next section.

IV. Coordinate Transformation to Simplify Bound Computation

In this section, we discuss a coordinate transformation to reduce the subdivision of the complex plane for QFT boundary calculation to one where we have symmetry that can be exploited to reduce computation. We transform the compensation to (x, y) using

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x_c \\ y_c \end{bmatrix} - \begin{bmatrix} x_{av} \\ y_{av} \end{bmatrix} \leftrightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_{av} \\ y_{av} \end{bmatrix} \tag{24}$$

The transformed region has as its origin the center of the Kharitonov rectangle as shown in Figure 3. Only the three regions where x and y are both positive need be considered because of the symmetry resulting from the transformation. For each point (x, y) on a boundary, symmetry yields the set of boundary points

$$\{(x, y), (-x, y), (x, -y), (-x, -y)\}$$

Transforming back to the original coordinates, we have the four points

$$\{(-x-x_{av}, -y-y_{av}), (x-x_{av}, -y-y_{av}), (-x-x_{av}, y-y_{av}), (x-x_{av}, y-y_{av})\}$$

We now give the equivalent of Theorem 1 for the transformed domain of (22).

Theorem 3. Given a transformed rectangular region D and a test point (x, y). Then the farthest point $\left(\hat{x}, \hat{y}\right)$ in D to (x, y) and the closest point $\left(\check{x}, \check{y}\right)$ in D to (x, y) are as given in Table 2 where $x_f = (x_+ - x_-)/2$ and $y_f = (y_+ - y_-)/2$.

Proof
The proof is similar to the proof of Theorem 1 and is omitted.

Table 2. Closest and Farthest Points for the Transformed Domain.

Region	$\left(\check{x}, \check{y}\right)$	$\left(\hat{x}, \hat{y}\right)$
I	(x, y_f)	$(-x_f, -y_f)$
II	(x_f, y_f)	$(-x_f, -y_f)$
III	(x_f, y)	$(-x_f, -y_f)$

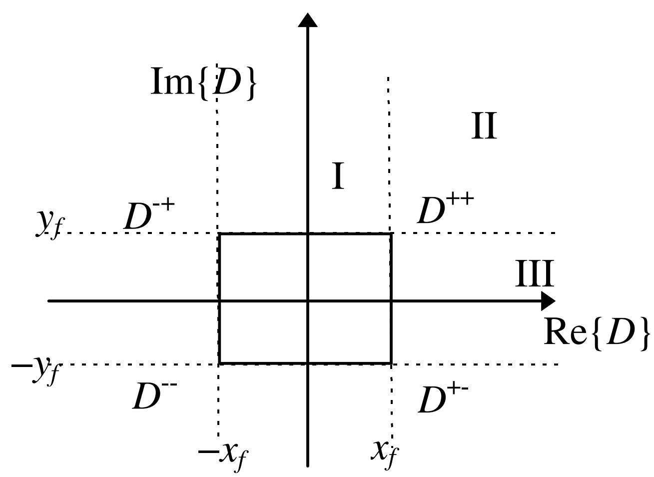

Fig. 3 The transformed Kharitonov rectangle.

We assume that we remain in a single region so that the points for maximum and minimum distances are fixed to derive the following monotonicity properties.

Theorem 4. As x or y vary in Region I or III, the rational function (1) is monotone increasing. In Region II, the behavior is unimodal.

Proof

To prove the Lemma, we simply differentiate the rational function of (14) and equate to zero to obtain the minima and maxima. For Region I, $\breve{x} = x$ and $\widehat{x} = -x_f$, so that (14) reduces to

$$R^2 = \frac{\left(x + x_f\right)^2 + \left(y + y_f\right)^2}{\left(y - y_f\right)^2} \le B^2$$

Differentiating with respect to x and equating to zero, we obtain

$$\tfrac{1}{2}\frac{\partial R^2}{\partial x} = \frac{x + x_f}{\left(y - y_f\right)^2} = 0$$

This indicates a positive derivative with the inadmissible negative root $x = -x_f$.

Differentiating with respect to y and equating to zero, we obtain

$$\tfrac{1}{2}\frac{\partial R^2}{\partial y} = \frac{\left(y + y_f\right)\left(y - y_f\right) - \left[\left(y + y_f\right)^2 + \left(x + x_f\right)^2\right]}{\left(y - y_f\right)^3} = 0$$

$$\left(x + x_f\right)^2 + 2y_f\left(y + y_f\right) = 0$$

This indicates a positive derivative with the inadmissible negative root

$$y = -y_f - \frac{\left(x + x_f\right)^2}{2y_f}$$

Because the derivatives are positive and have no positive real roots we conclude that the function is montone increasing.

Interchanging x and y, we prove monotonicity in Region III.

In Region II, $\breve{x} = x_f$ and $\widehat{x} = -x_f$, so that (14) becomes

$$R^2 = \frac{\left(x + x_f\right)^2 + \left(y + y_f\right)^2}{\left(x - x_f\right)^2 + \left(y - y_f\right)^2} \le B^2$$

$$\tfrac{1}{2}\frac{\partial R^2}{\partial x} = \frac{\left(x+x_f\right)\left[\left(x-x_f\right)^2+\left(y-y_f\right)^2\right]-\left(x-x_f\right)\left[\left(x+x_f\right)^2+\left(y+y_f\right)^2\right]}{\left[\left(x-x_f\right)^2+\left(y-y_f\right)^2\right]^2}$$

$$= \frac{-2x_f\left(x^2-x_f^2\right)-4yy_f x+2x_f\left[y^2+y_f^2\right]}{\left[\left(x-x_f\right)^2+\left(y-y_f\right)^2\right]^2} = 0$$

This yields the roots

$$x = -\frac{yy_f}{x_f} \pm \sqrt{\left(\frac{yy_f}{x_f}\right)^2 + y^2 + y_f^2 + x_f^2} \tag{25}$$

One root is negative and is therefore inadmissible. The second root is greater than x_f and is therefore inside Region 2. Therefore, the function is unimodal in Region 2. For large x the function approaches its minimum value of unity. We conclude that in Region II, the inadmissible region is in the vicinity of a maximum of the ratio R^2 as shown in Figure 4. Because the value of R^2 as $x \to x_f$ may be larger than B^2, the intersection of the curve of Figure 4 with the $R^2 = B^2$ line may occur at *one or two points*.

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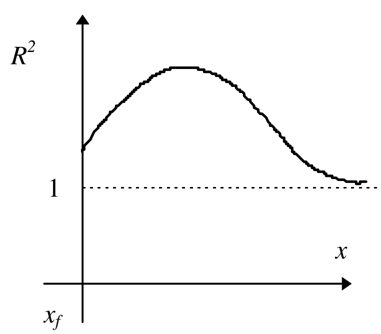

Fig.4 Unimodal function in Region II.

We now examine the boundaries of Theorem 2 for the transformed domain. The coordinate transformation is linear and the boundaries retain their form. However, the transformation reveals the following regarding the stability robustness boundary.

Theorem 5. The stability robustness boundary for the transformed domain in Regions I and II are straight lines parallel to the edges of the Kharitonov rectangle. In Region II, the boundary is a circle of radius *N*/*M* centered at (x_f, y_f).

Proof

The stability robustness bound is given by (20) with $(\breve{x}, \breve{y})$ defined for Regions I, II, III. In Region I, $(\breve{x}, \breve{y}) = \left(x, y_f\right)$ and the boundary is the vertical straight line

$$y = y_f + N/M$$

Similarly, in Region III, the boundary is the vertical line

$$x = x_f + N/M$$

In Region II, $(\breve{x}, \breve{y}) = \left(x_f, y_f\right)$ and the boundary is the circle

$$\left(y - y_f\right)^2 + \left(x - x_f\right)^2 = \left(N/M\right)^2$$

i.e. a circle of radius *N*/*M* centered at (x_f, y_f).

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Remark 2
Transforming the rectangle using the inverse transformation of (22) gives a second rectangle centerd at $(-x_{av}, -y_{av})$ with the symmetry revealed in the transformed domain holding in the vicinity of the rectangle. The boundaries and the permissible compensation can be easily plotted using this observation as demonstrated in the next section.

V. An Optimal, Linear Time Algorithm for Computing the Feasible Region of Compensation

We are now in a position to state the main algorithmic result of this paper. In (2) let the order of the numerator polynomial *N* be *m* and the order of the uncertain denominator polynomial *D* be *n*. Horner's Rule evaluates these polynomials in time $\theta(m)$ *resp.* $\theta(n)$ (Cormen *et al.*, [7] 1993, ex. 1.2-4). The numerator need be evaluated only once, but, in order to compute the Kharitonov rectangle, the denominator is recomputed (in optimal time $\theta(n)$) for each of n_w sample frequencies ω. For fixed ω, Theorem 2 prescribes how to determine, in optimal time $\theta(1)$, a set of constraints defining the feasible region of compensation. Thus, in time $\theta(n_w n + m)$, the frequency-divided boundary of minimum magnitude compensations can be computed. When, as is often the case, $n_w n$ is much greater than *m*, the running time is $\theta(n_w\ n)$. This compares favorably with the $\theta(n_w n_\gamma n_\phi n)$ running time of [5] and is certainly preferable to the $\theta(n_q n_w n_\gamma n_\phi n)$ running time of the traditional gridding approach [8], where n_q is the total number of grid points in the uncertain parallelpiped *Q*, n_γ is the number of grid points along the axis of compensation magnitude, and n_ϕ is the number of points along the compensation phase axis. Since the numerator need be evaluated only once, the Rule of Products implies that the search as described runs in time $\theta(n_q n_w n_\gamma n_\phi n + m)$.

To demonstrate the reduction in computational time achieved here we examine the case of a plant transfer function with six uncertain parameters and ten grid points for each parameter then $n_q = 10^6$. For $n_\gamma = n_\phi = 10$, the overall reduction achieved here is by a factor of one hundred relative to [5] and 10^8 relative to the gridding method!

V. Example: Position Control System

Consider the position control system with motor and load transfer function

$$G(s) = \frac{10}{s(\tau s + 1)}$$

where $\tau \in [\tau_{\min}, \tau_{\max}]$. For the compensator K, the product of the numerator of the transfer function and the compensator is $C = 10\ K$. The denominator is an interval polynomial whose value set at any frequency ω is the interval $[-\tau_{\max}\omega^2, -\tau_{\min}\omega^2]$ on the straight line parallel to the real axis with imaginary intercept ω. In this simple case, the twelve regions of Figure 2 reduce to 6 and the three regions of Figure 4 reduce to two. The bounds for the system are shown in Figure 5 together with the value set for a frequency of 1 rad/s.

For this example, c_{ry} is zero and the hyperbolas for Regions 7, 8, 11, 12 reduce to a single point outside the respective region. Regions 5, 6, 9, 10, are not defined for the example and need not be considered. In Regions 1-4, the tracking boundaries are circular arcs with the permissible region outside the arcs. The stability robustness boundary comprises circular arcs of radius N/M = 10/2 = 5 and straight lines parallel to the value set. The actuator bound is a circle of radius γ_{act}. The feasible region for compensation is inside the actuator boundary and outside both the tracking boundary and the stability robustness boundary.

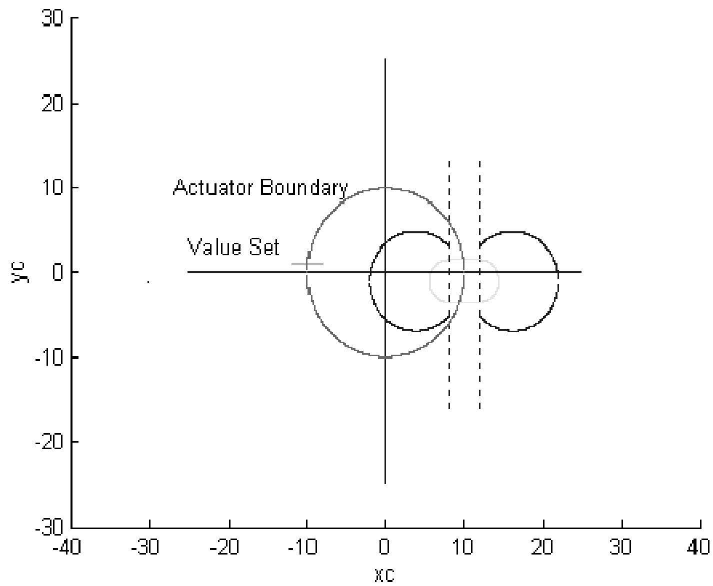

Fig. 5 QFT boundaries for the position control system.

VI. Conclusion

The simple case considered in this paper demonstrates the considerable computational saving possible in the computation of QFT bounds using value sets, coordinate transformations and subdivision of the compensator plane. Given a region where the farthest and closest points in the value set to a test point is determined, closed form expressions for the boundaries are obtained. This allows the computation of the QFT bounds in time $\theta(n_w\ n)$.

For a system with a numerator interval polynomial, a similar approach is possible with the distances sought being between two rectangular regions. For affine numerator and denominator polynomials, we seek the minimum and maximum distances between two polygons. These problems will be examined in more detail by the authors in future work.

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