On the Crossing Number of Tori

Antoine Bossard[1] Keiichi Kaneko[2] Frederick C. Harris Jr.[3]

[1]Graduate School of Science, Kanagawa University
2946 Tsuchiya, Hiratsuka, 259-1293 Japan

[2]Graduate School of Engineering, Tokyo University of Agriculture and Technology
2-24-16 Nakacho, Koganei, 184-8588 Japan

[3]Department of Computer Science and Engineering, University of Nevada, Reno
1664 N. Virginia Street, Reno 89557, NV, USA

ABSTRACT

Reducing the number of link crossings in a network to a minimum is a difficult problem which has nonetheless important applications. Circuit design with very-large-scale integration (VLSI) is an example of such an application. This problem is known as the crossing number problem. Finding a general solution to this problem has been shown to be NP-hard. Hence, solutions for particular classes of graphs have been proposed. In this paper, we focus on tori as they have proven very popular as interconnection network of massively parallel systems; see for instance the Fujitsu K and Cray Titan supercomputers. We start by devising an optimal upper bound on the crossing number of a two-dimensional k-ary torus. This first result is extended to obtain an upper bound on the crossing number of a three-dimensional k-ary torus. Finally, we derive from these discussions an upper bound on the crossing number of a k-ary n-dimensional torus. The proposed bounds and drawing methods are empirically evaluated with experiments involving system-generated torus drawings and the automatic calculation of their crossing numbers.

KEYWORDS

VLSI; interconnect; network; intersection; circuit;

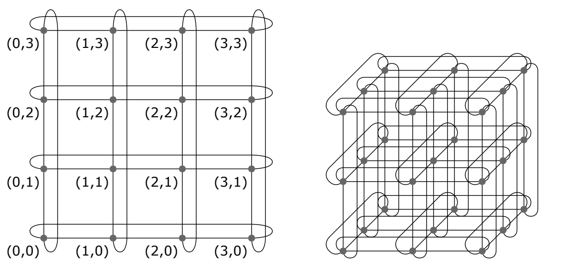
1 INTRODUCTION

Informally, the crossing number of a graph is the minimum number of edge crossings when drawing the graph on a surface. This graph drawing problem has important applications in various domains: circuit design (VLSI) [1, 2, 3] and graph visualisation [4] are two examples of such applications. This problem is famous for its complexity. Indeed, it has been shown to be NP-hard [5]. General solutions for the crossing number problem have been notably studied by Erdős and Guy [6], Turán [7], and somewhat more recently in [8]. Besides, an algorithm for calculating the crossing number (i.e., an optimal solution to the crossing number problem) has been proposed in [9].

Given the NP-hard complexity of this problem, it has been solved for special classes of graphs. For instance, Sýkora and Vrťo focused on hypercubes [10]. More recently, new results have been obtained by Faria et al. for this same class of graphs (hypercubes and their variants) [11]. Other examples of specific classes of graphs for which the crossing number problem has been studied include complete graphs [12, 13] and stars [14].

In this paper, we focus on such a class of graphs: tori. Effectively, as shown for instance by the TOP500 ranking, many major supercomputers such as the Fujitsu K, the Cray Titan and IBM Blue Gene/P are relying on the torus topology for their interconnection network. Tori are very popular as interconnection network of massively parallel systems for their simplicity and scalability properties, amongst others [15, 16].

The rest of this paper is organised as follows. Important notations, definitions and results used throughout this paper are recalled in Section 2. The crossing number of a two-dimensional k-ary torus is then discussed in Section 3, and that of a three-dimensional k-ary torus is discussed in Section 4. Next, an upper bound on the crossing number of an n-dimensional k-ary torus is derived in Section 5. Preliminary empirical evaluation of the pro-



(a) (b)

Figure 1. (a) A 2-dimensional 4-ary torus $T(2,4)$. (b) A 3-dimensional 3-ary torus $T(3,3)$.

posed method is conducted with experiments in Section 6. Finally, this paper is concluded in Section 7.

2 PRELIMINARIES

We recall in this section several definitions and notations used throughout this paper. First, the definition of the torus network topology is recalled.

Definition 1. *An n-dimensional k-ary torus $T(n,k)$ is an undirected graph whose vertices consist in the k^n n-vectors induced by the set $\{0,1,\ldots,k-1\}^n$. Furthermore, two nodes $u=(u_1,u_2,\ldots,u_n)$ and $v=(v_1,v_2,\ldots,v_n)$ of a $T(n,k)$ are adjacent if and only if $\exists j$ $(1 \leq j \leq n)$ such that $\forall i$ $(1 \leq i \leq n,\ i \neq j)$ $u_i = v_i$ and $u_j = v_j \pm 1 \pmod k$.*

A 2-dimensional 4-ary torus $T(2,4)$ and a 3-dimensional 3-ary torus $T(3,3)$ are illustrated in Figure 1.
Regarding the notations used hereinafter, most are standard and thus self-explanatory, and the number of vertices (a.k.a. nodes) and edges of a graph G are denoted by $|G|$ and $||G||$, respectively.
Next, several definitions and results relating to the crossing number of a graph are recalled. First and foremost, a *point* refers hereinafter to a geometrical coordinate; it should thus not be confused with a graph vertex. One graph vertex induces one point.
A *drawing* of a graph G is the representation of its vertices and edges on a surface (typically, a plane). It connects vertices with arcs corresponding to the edges of G; the two endpoints of an arc thus correspond to two adjacent vertices. *Regions* correspond to the complement of the union of the arcs (i.e., the points making the arcs) of a drawing of G.
An *embedding* of a graph G on a surface S is a drawing of G on S such that any two arcs may intersect only at the point corresponding to the vertex to which they are both incident. If S is a plane, then the graph induced by such an embedding is called a *plane graph*.
A region is a *2-cell* if and only if any closed curve contained by the region can be continuously contracted to a single point. If all the regions of an embedding are 2-cells, such an embedding is called a *2-cell embedding*.
We can now recall the well-known Euler formula [17] in the below theorem.

Theorem 1 (Euler's formula)**.** *Let G be a connected graph of n vertices, m edges and with a 2-cell embedding of r regions. Then,*

$$n - m + r = 2$$

Considering a drawing in the plane of a graph, a *crossing* is a point included by exactly two distinct arcs and that is not an endpoint of both arcs. A crossing is thus induced by one pair of arcs. Note that any two distinct arc pairs each inducing a crossing result into two crossings, and this even if the two crossings are the same point in the plane.

Definition 2. *The crossing number $cr(G)$ of a graph G is the minimum number of crossings amongst the drawings of G in the plane.*

Definition 2 directly induces that a graph G satisfies $cr(G) = 0$ if and only if it is a plane graph.

3 ON THE CROSSING NUMBER OF A $T(2,k)$

In this section, we give a constructive proof for an upper bound on the crossing number of $cr(T(2,k))$. This construction of a $T(2,k)$ in three steps is illustrated in Figure 2.
First, the corresponding 2-dimensional k-ary mesh is considered. A mesh is by definition

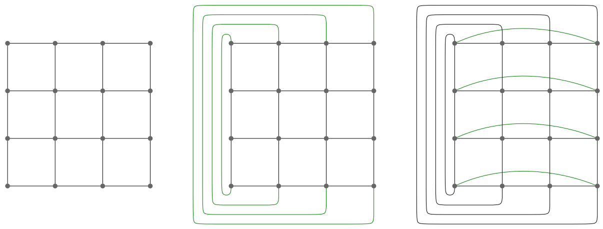


Figure 2. Illustrating the proposed $T(2,k)$ construction method in three steps (1) to (3) with a $T(2,4)$, inducing an upper bound on $cr(T(2,k))$.

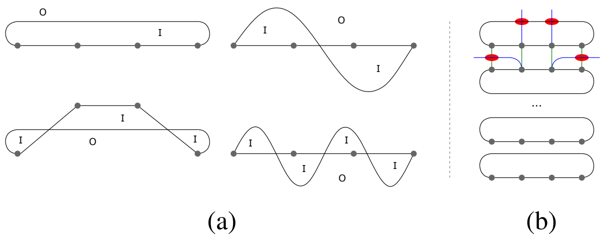


Figure 3. (a) Illustrating the crossings induced by a ring and a few of its isomorphisms when $k = 4$: 0, 1, 2 and 3 crossings are induced. (b) Connection of nodes of different rings.

a planar graph; it has $(k-1)^2+1$ regions ($+1$ to count the outside, unbounded region).
Second, the wrap-around edges of one dimension of the torus are drawn such that they do not induce any crossing. These k new edges induce k new regions; the number of regions in the obtained graph is now $(k-1)^2+1+k$. The graph is indeed still planar as assured by Theorem 1: $k^2-[2k(k-1)+k]+[(k-1)^2+1+k] = 2$.
Third, the wrap-around edges of the remaining dimension are added. Each such new edge crosses at least $k-2$ edges and thus induces at least $k-2$ crossings. Therefore, in total, the minimum number of crossings in such a drawing of a $T(2,k)$ is equal to $k(k-2)$.

Theorem 2. *The crossing number of a $T(2,k)$ is as follows:*

$$cr(T(2,k)) = k(k-2)$$

Proof. We have already given a constructive proof that shows the relation

$$cr(T(2,k)) \leq k(k-2)$$

We next show that this upper bound on the crossing number of $T(2,k)$ is tight.
A $T(2,k)$ torus consists of k rings, each made of k nodes (i.e., a k-ring). More generally, a $T(2,k)$ torus consists of k subgraphs that are each isomorphic to a k-ring. Hence, depending on the location of the k nodes, zero or more crossings are induced for each k-ring; refer to Figure 3a. The nodes of such k-rings can be connected by going through the inside of the ring or not (i.e., staying outside of the ring). Areas inside and outside a ring are indicated in Figure 3a with "I" and "O" letters, respectively. Except the two nodes at both extremities of the ring (obviously, this can be any two nodes), it is necessary so as to connect nodes of distinct rings to either go through the inside of the ring, which thus induces at least one crossing with the ring, or to go through the outside of the ring, which thus induces at least one crossing with an edge connecting two nodes of different rings. See Figure 3b; crossings in both situations are highlighted in red.
Therefore, given that there are k such rings and that the previously described crossing situation occurs at $k-2$ nodes on each ring, we can deduce that a $T(2,k)$ includes at least $k(k-2)$ crossings. This number is obviously increased if, as shown in Figure 3a, rings themselves include crossings, or if the rings have some overlaps.
Finally, if there are r (≥ 0) concentric ring clusters, say with each ring cluster made of $c_1, c_2, \ldots, c_r$ rings, connection of the nodes of the innermost rings induces in total at least $(c_1+c_2+\ldots+c_r)(k-2)$ crossings. Hence, at least $k(k-2)$ crossings are induced in total for cluster rings and non-cluster rings.

□

4 ON THE CROSSING NUMBER OF A $T(3,k)$

In this section, we give a constructive proof for an upper bound on the crossing number $cr(T(3,k))$. An illustration of the proposed $T(3,k)$ drawing method is given in Figure 4.
For convenience, the edges of a sub-torus

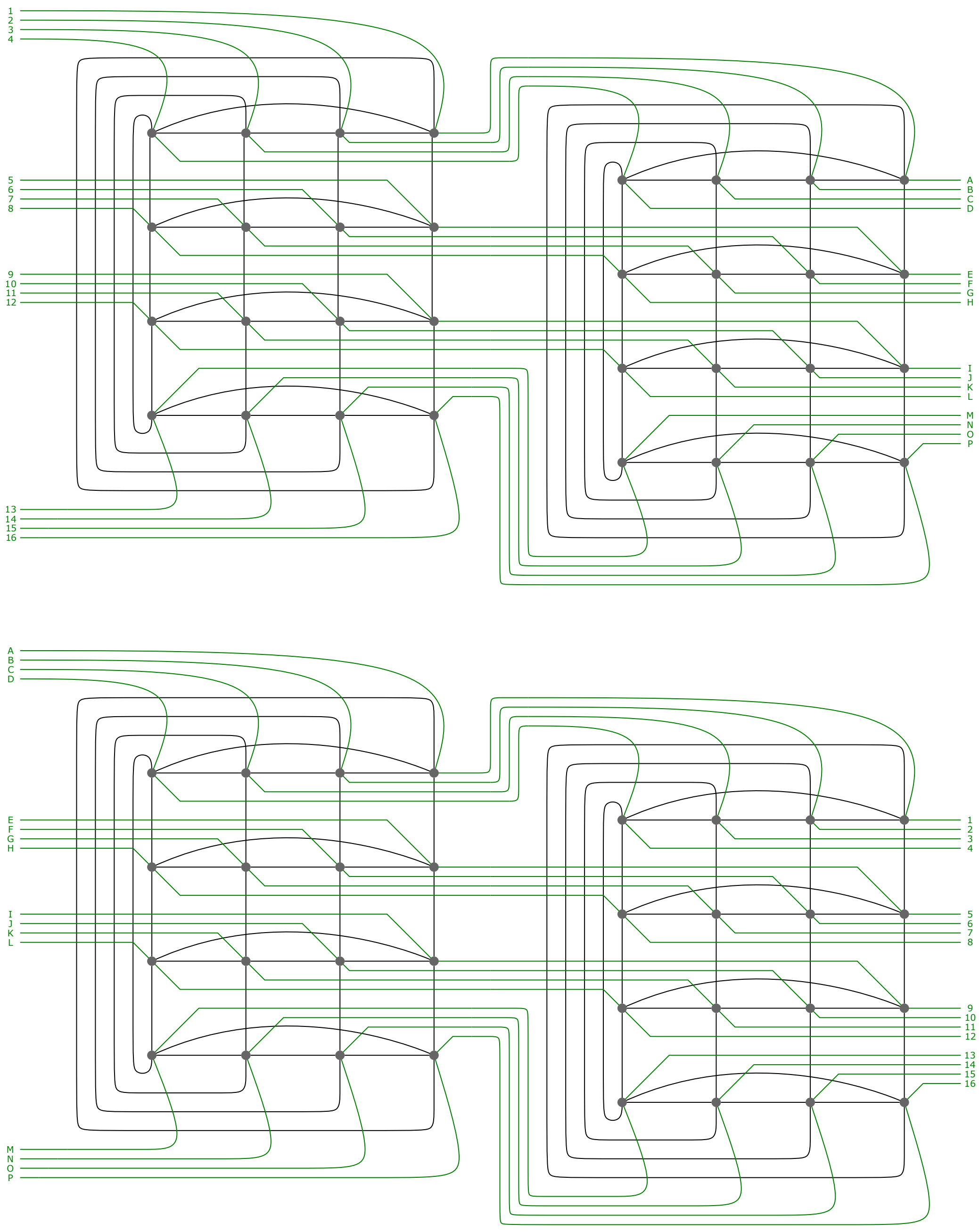


Figure 4. Illustrating the proposed upper bound on $cr(T(3,k))$ with a $T(3,4)$. The 1–16 and A–P edges are cut for disposition matters only; they induce no additional crossing.

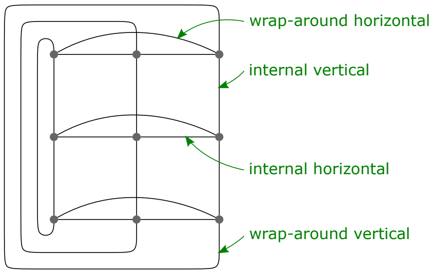


Figure 5. Naming sub-torus edges for the sake of readability.

$T(2,k)$ are named according to four categories: *wrap-around horizontal*, *wrap-around vertical*, *internal horizontal* and *internal vertical*. This naming scheme is detailed in Figure 5.

4.1 $T(3,k)$ construction

Our torus construction approach relies on the recursive property of a torus; precisely, a $T(n,k)$ torus consists of k sub-tori $T(n-1,k)$. Thus, the proposed drawing of a $T(3,k)$ is based on k drawings of a $T(2,k)$ as per Section 3. Hence, the number of crossings in such a drawing of a $T(3,k)$ is as follows:

$$cr(T(3,k)) \leq k \cdot cr(T(2,k)) + \alpha$$

with α the number of crossings induced by the edges of the third dimension (a.k.a. the new edges, i.e., the edges connecting nodes of different sub-tori). It should be noted that even if the drawing of a $T(2,k)$ as per Section 3 is optimal with respect to the crossing number, there is no guarantee that an optimal drawing of a $T(3,k)$ can be derived from such optimal $T(2,k)$ drawings. In other words, we have the relation $cr(T(3,k)) \leq k \cdot cr(T(2,k)) + \alpha$ as stated previously, but it remains to be proved that $cr(T(3,k)) = k \cdot cr(T(2,k)) + \alpha$ holds.
In practice, the proposed construction arranges the k drawings of sub-tori $T(2,k)$ horizontally one beside the other. For the sake of clarity, the third dimension edges (a.k.a. new edges), drawn in green hereinafter, are categorised as either IN or OUT edges depending on whether they connect a vertex from the left (IN edges) or from the right (OUT edges) of a sub-torus. Inside each sub-torus $T(2,k)$, the vertices are classified according to their vertical position y: a vertex is thus of class $y = i$ with $1 \leq i \leq k$. By convention, the class $y = 1$ specifies the vertices at the top of the sub-torus.
Depending on the y position of sub-torus nodes, three cases are distinguished to draw the third dimension edges.

Case $y = 1$
This is a special case. The IN edges connect vertices from above such that they do not cross each other, and avoiding wrap-around vertical edges as much as possible. The OUT edges connect vertices from below such that they do not cross each other when connecting the vertices of the next sub-torus.

Case $2 \leq y \leq k-1$
This is the general case. The IN edges connect vertices from above such that they do not cross each other The OUT edges connect vertices from below such that they do not cross each other when connecting the vertices of the next sub-torus.

Case $y = k$
This is a special case. The IN edges connect vertices from below such that they do not cross each other, and avoiding wrap-around vertical edges as much as possible. The OUT edges connect vertices from above such that they do not cross each other when connecting the vertices of the next sub-torus.

With such a connection scheme, we are able to guarantee that the new edges do not cross each other. In Figure 4, the 1–16 and A–P edges are cut for disposition matters only; it is easy to see that they induce no additional crossing since the 1–16 and A–P edge sequences are in the same order.

4.2 Counting the number of induced crossings

The objective of this section is to count the number of crossings induced by the third dimension edges, that is calculating the value of

α. We first count such a number of crossings inside one sub-torus $T(2, k)$, and then multiply this number by k the number of sub-tori to eventually obtain α.
The number of crossings induced by the third dimension edges inside one sub-torus is established by distinguishing as in Section 4.1 the three vertex classes $y = 1, 2 \leq y \leq k-1$ and $y = k$. Let $\#(C)$ be the number of crossings induced by the new edges inside one sub-torus for all the nodes of class C. So, the number of crossings β induced by the new edges inside one sub-torus is as follows:

$$\beta = \#(y = 1) + \#(2 \leq y \leq k-1) + \#(y = k)$$

Thus, the total number of crossings induced by the new edges α is as follows:

$$\alpha = k \cdot \beta$$

The counting $\#(C)$ for the three vertex classes C is detailed below. The sums from 1 to k are used to iterate the horizontal position of vertices inside the sub-torus. For each vertex class, the crossing count is realised separately for the IN and OUT edges.

Calculation of $\#(y = 1)$

The number of crossings induced by the IN edges:

$$\underbrace{\left[\sum_{i=1}^{k}(k-i)\right]}_{\text{wrap-around vertical}} + \underbrace{(k-2)}_{\substack{\text{wrap-around}\\ \text{horizontal}}}$$

The number of crossings induced by the OUT edges:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}}$$

Hence,

$$\begin{aligned}
&\#(y = 1)\\
&\quad = \left[\sum_{i=1}^{k}(k-i)\right] + (k-2) + \sum_{i=1}^{k}(k-i)\\
&\quad = k^2 - 2
\end{aligned}$$

Calculation of $\#(2 \leq y \leq k-1)$

The number of crossings induced by the IN edges:

$$\underbrace{k^2}_{\substack{\text{wrap-around}\\ \text{vertical}}} + \underbrace{(k-2)}_{\substack{\text{wrap-around}\\ \text{horizontal}}} + \underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}}$$

The number of crossings induced by the OUT edges:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}}$$

Hence,

$$\begin{aligned}
&\#(2 \leq y \leq k-1)\\
&\quad = \underbrace{(k-2)}_{\substack{\text{for each } y\\ \text{in } 2 \leq y \leq k-1}} \cdot \Bigg[k^2 + (k-2)\\
&\qquad + \sum_{i=1}^{k}(k-i) + \sum_{i=1}^{k}(k-i)\Bigg]\\
&\quad = 2k^3 - 4k^2 - 2k + 4
\end{aligned}$$

Calculation of $\#(y = k)$

The number of crossings induced by the IN edges:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{wrap-around vertical}}$$

The number of crossings induced by the OUT edges:

$$\underbrace{\left[\sum_{i=1}^{k}(k-i)\right]}_{\text{internal vertical}} + \underbrace{(k-2)}_{\substack{\text{wrap-around}\\ \text{horizontal}}}$$

Hence,

$$\begin{aligned}
&\#(y = k)\\
&\quad = \sum_{i=1}^{k}(k-i) + \sum_{i=1}^{k}(k-i) + (k-2)\\
&\quad = k^2 - 2
\end{aligned}$$

As a result, we have

$$\beta = 2k^3 - 2k^2 - 2k$$

and

$$\alpha = 2k^4 - 2k^3 - 2k^2$$

This discussion on an upper bound for $cr(T(3,k))$ is summarised in the following theorem.

Theorem 3. *The crossing number of a $T(3,k)$ satisfies the following relation:*

$$cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2$$

Proof. This can be easily derived from the previously established expression $cr(T(3,k)) \leq k \cdot cr(T(2,k)) + \alpha$. □

5 DERIVING AN UPPER BOUND ON THE CROSSING NUMBER OF A $T(n,k)$

We advance this discussion by deriving an upper bound on the crossing number of a $T(n,k)$ from the previously established upper bound on the crossing number of a $T(3,k)$. Providing a tight estimation is difficult; we proceed simply as follows and aim at refining the obtained bound in future works.

Because it is impractical to derive an upper bound on the crossing number of a $T(n,k)$ by a method similar to that of Section 4, we derive one as follows: for each n-th dimension edge (a.k.a. new edge) count amply the maximum number of crossed edges in one sub-torus $T(n-1,k)$, then multiply this number by the number of n-th dimension edges in one sub-torus, and finally multiply by the total number of sub-tori $T(n-1,k)$. In this approach, we assume that the n-th dimension edges do not cross each other. The validity of this assumption is explained later.

In other words, the idea is to maximise the number of crossings per n-th dimension edge, and to define recursively an upper bound on the crossing number of a $T(n,k)$. It should be noted that in this approach we derive a drawing of a $T(n,k)$ from several $T(n-1,k)$ sub-tori, even though it remains to be proved that the number of crossings in a $T(n,k)$ is minimal when reusing $T(n-1,k)$ sub-tori as is. This is expressed formally below.

First, $\mathcal{C}$ the maximum number of crossings induced by n-th dimension edges in a $T(n,k)$ is defined as follows:

$$\mathcal{C} \quad \leq \quad \underbrace{k}_{\substack{\text{for each}\\ \text{sub-torus}}} \cdot \underbrace{2k^{n-1}}_{\substack{\text{for each node}\\ \text{2 new edges}}} \cdot \underbrace{\delta}_{\substack{\text{max \# of}\\ \text{crossings for}\\ \text{1 new edge}}}$$

where δ the maximum number of crossings for one n-th dimension edge in one sub-torus $T(n-1,k)$ can be obviously safely defined as

$$\delta = ||T(n-1,k)|| = (n-1)k^{n-1}$$

Therefore, relying on Theorem 3, an upper bound is recursively deduced as follows:

$$\begin{aligned} cr(T(2,k)) &\leq k(k-2) \\ cr(T(3,k)) &\leq 2k^4 - k^3 - 4k^2 \\ cr(T(n,k)) &\leq k \cdot cr(T(n-1,k)) + \mathcal{C} \\ &\leq k \cdot cr(T(n-1,k)) \\ &\quad + 2(n-1)k^{2n-1} \end{aligned}$$

Regarding the assumption that the n-th dimension edges do not cross each other, it is indeed all right to assume so since these edges can be drawn in a similar fashion as for a $T(3,k)$ as illustrated in Figure 6. The dimension (i.e., value of n) does not matter; as stated in Section 4, since we take care to retain the IN and OUT edge order of the new edges, they can be connected without additional crossing. So, it is sound to assume that each n-th dimension edge crosses at most $\delta = ||T(n-1,k)|| = (n-1)k^{n-1}$ edges inside one sub-torus $T(n-1,k)$. The above discussion is summarised in the following theorem.

Theorem 4. *The crossing number of a $T(n,k)$ satisfies the following relation:*

$$cr(T(n,k)) \leq k \cdot cr(T(n-1,k)) + 2(n-1)k^{2n-1}$$

6 PRELIMINARY EXPERIMENT

We have conducted a preliminary empirical evaluation to experimentally validate the proposed method and the corresponding quantitative results.

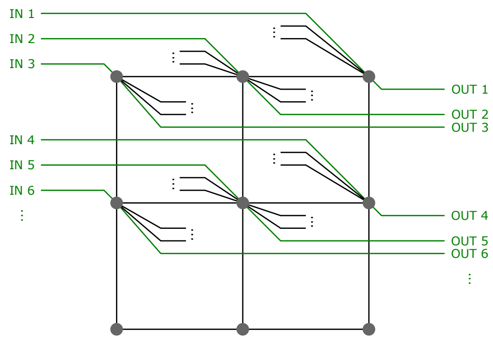


Figure 6. Drawing the n-th dimension edges (in green) so that they do not cross each other. Several edges are abbreviated for clarity.

To this end, we have implemented a function that calculates the minimum number of crossings in a graph given a natural number g that defines a $g \times g$ square grid. This grid is used for the repartition of graph vertices. The larger the value of g, the higher number of possible graph drawings. Because this parameter g can be infinitely large, it suffices to avoid considering curved edges (i.e., edges are drawn with straight lines only).

The initial implementation of this function satisfies total correctness as, given a g, it processes each of all the grid position permutations, one by one, each vertex being assigned to one such grid position. Since the number of permutations is obviously finite, this program eventually terminates, returning the minimum number of crossings found. Yet, this exhaustive approach takes much time and becomes rapidly impracticable as the order of the input graph increases. Hence, we have relied for this experiment on a second implementation, this time satisfying only partial correctness, that instead of going through all the grid position permutations processes random ones. The algorithm terminates if a graph drawing with no crossing is found, but in practice, the intermediary values of the minimum number of crossings are used as experimental data. The results of this experiment are measured as follows: if the current number of crossings is smaller than the previously stored number of crossings, the former replaces the latter, and this value is output together with the total time taken until finding the corresponding graph drawing. The

Table 1. The minimum number of crossings in a $T(2,3)$ as obtained from the experimentation.

	Elapsed time (in seconds)	Minimum number of crossings
$g = 5$	0	40
	0.001	23
	0.002	17
	0.003	15
	0.004	14
	0.005	12
	0.007	10
	0.015	4
	1.496	3

pseudo-code of this function is given in Algorithm 1.

Algorithm 1 randomized(g, n, k)

```
Input: The grid side g, a dimension n and an arity k,
  inducing a T(n, k).
  p ← (0 1 2 ... g^2 − 1) // initial vertex positioning
  a ← +∞ // result accumulator
  while a > 0 do
    Coord2D coords[k^n] // coordinate from position index
    for i ← 0 to k^n − 1 do
      coords[i] ← (⌊p_i/g⌋, p_i mod g)
    end for
    c ← crossings(coords) // get the number of crossings
    if c < a then
      a ← c
      print c // the current minimum number of crossings
    end if
    p ← shuffle p
  end while
```

First, the results obtained in the case of a 2-dimensional 3-ary torus $T(2,3)$ are given in Table 1. The crossing number as established in Theorem 2, that is 3, is obtained in this experiment as well (the program lastly outputs 3, and continues to search for a smaller crossing number, which exists not as proven), thus experimentally confirming Theorem 2 for a $T(2,3)$.
Second, the experimental results obtained in the cases of a $T(2,4)$ and of a $T(3,3)$ are respectively given in Figures 7 and 8. The upper bound on the crossing number as calculated from the proposed method is also plotted for reference (labelled "proposed"). In one plot, the results obtained with several grid sizes (i.e., different values of g) are given.
As plotted, the theoretical upper bound on the

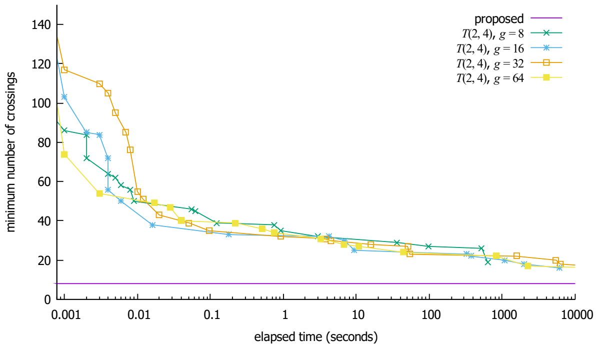


Figure 7. The minimum number of crossings in a $T(2,4)$ as obtained from experimentation.

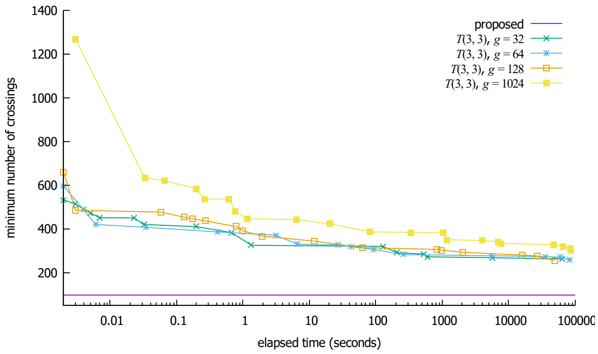


Figure 8. The minimum number of crossings in a $T(3,3)$ as obtained from experimentation.

crossing number of a $T(2,4)$ is 8, and that of a $T(3,3)$ is 99. Hence, the obtained empirical results show that our constructive proof of a torus provides a non-trivial upper bound on the crossing number of a torus as experimentally considering random graph drawings tends towards, but remains at some distance from, these established upper bounds (and the corresponding drawings). Besides, it can also be noticed especially on Figure 8 that the size of the grid used to draw graphs ought to be carefully devised: it should be neither too small, in which case the graph vertex arrangement granularity would be too big, nor too large, in which case the running time required would explode due to the increased number of possible vertex repartitions.

7 CONCLUSIONS

Finding the crossing number of a graph is a problem which has important applications in domains such as circuit design (VLSI) and graph visualisation. Deriving a general solution to this problem has been shown to be NP-hard. Hence, solutions are given for specific classes of graphs. In this paper, we focus on tori for that they are very popular as interconnection network of massively parallel systems. We have first derived an optimal upper bound on the crossing number of a two-dimensional k-ary torus. Then, we have extended this discussion to obtain an upper bound on the crossing number of a three-dimensional k-ary torus. Precisely, we have shown that $cr(T(3,k))$ the crossing number of a $T(3,k)$ satisfies $cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2$. Finally, we have derived from these results an upper bound on the crossing number of a k-ary n-dimensional torus. Precisely, we have shown that $cr(T(n,k))$ the crossing number of a $T(n,k)$ satisfies $cr(T(n,k)) \leq k \cdot cr(T(n-1,k)) + 2(n-1)k^{2n-1}$. The proposed bounds and the corresponding drawing methods have been empirically evaluated with several experiments involving the automatic calculation of the crossing number of system-generated torus drawings.
Regarding future works, the refining of the proposed upper bounds is one meaningful objective. In addition, improving the efficiency of the algorithm used for the experiment is another interesting work as it could lead to even more assertive results regarding the established upper bounds.

ACKNOWLEDGEMENTS

The authors sincerely thank the reviewers for their insightful comments. This research was partly supported by a research fund of the Grant-in-Aid for Scientific Research (C) of the Japan Society for the Promotion of Science under Grant no. 17K00093.

REFERENCES

[1] F. T. Leighton, Complexity issues in VLSI, MIT Press, Cambridge, MA, USA, 1983.

[2] F. T. Leighton, “New lower bound techniques for VLSI”, Math. Systems Theory, vol. 17, no. 1, pp. 47–70, 1984.

[3] O. Sýkora and I. Vrťo, “On VLSI layouts of the star graph and related networks”, Integration, the VLSI Journal, vol. 17, no. 1, pp. 83–93, 1994.

[4] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis, Graph drawing: algorithms for visualizations of graphs, Prentice Hall, New York, NY, USA, 1999.

[5] M. R. Garey and D. S. Johnson, “Crossing number is NP-complete”, SIAM Journal on Algebraic and Discrete Methods, vol. 4, no. 3, pp. 312–316, 1983.

[6] P. Erdős and R. K. Guy, “Crossing number problems”, American Mathematical Monthly, vol. 80, no. 1, pp. 52–58, 1973.

[7] P. Turán, “A note of welcome”, Journal of Graph Theory, vol. 1, no. 1, pp. 7–9, 1977.

[8] J. Pach, J. Spencer, and G. Tóth, “New bounds on crossing numbers”, Discrete Computational Geometry, vol. 24, no. 4, pp. 623–644, 2000.

[9] F. C. Harris, Jr. and C. R. Harris, “A proposed algorithm for calculating the minimum crossing number of a graph”, in Proceedings of the Eighth International Conference on Graph Theory, Combinatorics, Algorithms, and Applications, vol. 2, pages 469–478. Western Michigan University, June 1998.

[10] O. Sýkora and I. Vrťo, “On crossing numbers of hypercubes and cube connected cycles”, MPI-I-91-124, Max-Planck-Institut für Informatik, Saarbrücken, Germany, 1991.

[11] L. Faria, C. Miraglia Herrera de Figueiredo, O. Sýkora, and I. Vrťo, “An improved upper bound on the crossing number of the hypercube”, Journal of Graph Theory, vol. 59, no. 2, pp. 145–161, 2008.

[12] J. R. Fredrickson, On the crossing number of complete graphs: growing minimal K_n from minimal K_{n-1}, Ph.D. dissertation, University of Nevada Reno, 2006.

[13] L. S. Humphrey, Efficient generation of minimal graphs using independent path analysis, Master thesis, University of Nevada Reno, 2006.

[14] M. Klešč and Š. Schrötter, “On the crossing numbers of Cartesian products of stars and graphs of order six”, Discussiones Mathematicae Graph Theory, vol. 33, no. 3, pp. 583–597, 2013.

[15] K. Kaneko and A. Bossard, “A set-to-set disjoint paths routing algorithm in tori”, International Journal of Networking and Computing, vol. 7, no. 2, pp. 173–186, 2017.

[16] A. Bossard and K. Kaneko, “Torus-Connected Cycles: a simple and scalable topology for interconnection networks”, International Journal of Applied Mathematics and Computer Science, vol. 25, no. 4, pp. 723–735, 2015.

[17] G. Chartrand, L. Lesniak, and P. Zhang, Graphs & digraphs (sixth edition), Chapman and Hall/CRC, Boca Raton, FL, USA, 2015.