An Optimization Algorithm for the Sale of Overage Data in Hong Kong's Mobile Data Exchange Market

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***Abstract*—Internet service providers are offering shared data plans where multiple users may purchase and share a single pool of data. In the Chinese economy, users have the ability to sell unused data on the Hong Kong Exchange Market, called "2cm", currently maintained by AT&T internet services. We propose a software-defined network for modeling this wireless data exchange market; a fully connected, pure "point of sale" market. A game-theoretical analysis identifies and defines rules for a progressive second price (PSP) auction, which adheres to the underlying market structure. We allow for a single degree of statistical freedom – the reserve price – and show that data exchange markets allow for greater flexibility in acquisition decision-making and mechanism design with an emphasis on optimization of software-defined networks.**

We have designed a framework to optimize this strategy space using the inherent elasticity of supply and demand. Using a game theoretic analysis, we derive a buyer-response strategy for wireless users based on second price market dynamics and prove the existence of a balanced pricing scheme. We examine shifts in the market price function and prove that the desired properties for optimization to a Nash equilibrium hold.

keywords: software-defined networks, mobile share, game theory, second-price auction

I. Introduction

Mobile data usage is quickly outpacing voice and SMS in wireless networks. Multi-device ownership has led to the introduction of the shared data plan [1]. Using an account service, users are able to keep track of data usage in real time across all their devices. The shared data service plan requires that users hold an a priori knowledge of demand and supply with respect to their data plan in order to form a strategy, meaning that a user must *plan* to either buy or sell thier overage data. In our formulation, we address several topics: data as a product in the real-monetary market, and data as network resource in a wireless topology.

Many new services are found exclusively on mobile devices. Companies are moving their software from (wired) grid-based to node-based communication. For example, the move from a standard website to a mobile phone app. Software-defined networking (SDN) addresses the new environment of wireless communication devices, allowing for a programmable network architecture. The account services that manage wireless shared data plans decentralize network management, and mobility becomes a factor in SDN design. Individual mobile devices provide flexibility, and may make decisions regarding local network infrastructure. There is a clear need for algorithms designed for optimization in this space. In many cases, the direct communication between mobile devices allows for a simple mutation of classic optimization models. Auctions are key in SDN for the fair allocation of resources. For this work, we focus on mobile data, an infinitely divisible and distributable quantity. Mobile data represents online data accessed using a wireless network. In [2], Lazar and Semret introduced the Distributed Progressive Second Price Mechanism (PSP) for bandwidth allocation. Such an auction is (1) easily distributed, and (2) allocates an infinitely divisible resource. A PSP auction is defined as distributed when the allocations at any element depend only on local state; no single entity holds a global market knowledge. We consider the multi-auction: where each auctioneer is a user selling data to their peers.

The model for data exchange was recently adopted by China Mobile Hong Kong (CMHK), who released a platform, called 2cm (secondary exchange market), creating a secondary market where users can buy and sell data from each other. CMHK owns and moderates 2cm, where CMHK the only auctioneer, and computes allocations of mobile data based on bids submitted to the platform. We focus on providing users with an incentive framework so rational users will choose a collaborative exchange. This collaborative exchange is the (built-in) transformation from the direct-revelation mechanism (truthful bidding) to the desired message space (actual bids).

We describe our auction mechanism as a pure-strategy progressive game with incomplete, but perfect information. The market strategy is determined by the impact of user behavior on market dynamics. The optimal objective is defined as a rational user's valuation of digital property. In classic mechanism design, with multiple user types, there is no single way to design the transformation from the direct revelation mechanism to its corresponding computational design. As in [2], our incentive for a user to truthfully reveal its type is built into the user strategies. We determine (at least one) local equilibrium is a result of incentive compatibility (truthfulness) in strategic bidding, and so our formulation holds the desired PSP qualities. Our derivation of strategies depend on the ratio of supply and demand, and consequently, on the ratio of buyers to sellers.

This is the first work to provide a comprehensive derivation of an auction mechanism with respect to the CMHK platform. The rest of of this paper is structured as follows: Section II presents the related work on auction theory and resulting policy software. Section III details the mathematical structure of the data-exchange market, which we present as an extension of the market in [2]. The analysis of user behavior and the resulting algorithms are presented in Section IV along with a simple example. Conclusions and Future Work follow in Section V.

II. Related Work

Progressive second price auctions are used for optimal allocation in a variety of scenarios, and for different reasons. Different definitions of social welfare define different strategies. Typical goals of optimization are the maximization of revenue, and optimal allocation. Other papers focus, taken from auction theory, optimize seller's reserve prices, or market price. Results derived from game theory focus on player strategy, as in this work. In [3], user strategy gives a "quantized" version of PSP, improving the rate of convergence of the game. Modifications to the mechanism that result in improved convergence also appear in [4], which relies on an approximation of market demand. Another mechanism derived from game theory [5], derives optimal strategies for buyers and brokers (sellers), and further shows the existence of network-wide market equilibria by representing the market dynamics as a system of equations.

Allowing a user preference to, loosely, represent a policy, we may interpret a the rules of the data exchange market as a policy scheme, where the ISP is assumed to enforce the rules and the market dynamics play out as a game among "users" of the game. So in a distributed system, users are allowed to set their own policies, and the ISP is responsible for implementing the framework to support their preferences. Trusted management systems are based on the Common Information Model (CIM), and focus on policy-based management, for example the "Policy-Maker" toolkit. In general, the translation of policy-based management systems to SDN focuses on combining the simplicity of policy-based implementation with the flexibility of SDN, as in the meta-policy system, CIM-SDN [6].

Game-theoretical analysis of mobile data has been presented in [7] as a framework for mobile-data offloading. In our analysis, the stability of the game is expressed as the set of equilibria, or fixed points, of the system. When considering the distributed and decentralized allocation of resources, a variety of equilibria exist for heterogeneous and homogeneous services once a certain set of conditions is met, one of which is truthfulness.

III. The Market Mechanism

In a distributed PSP auction, the design must must meet a certain set of known criteria: (1) *truthfulness* (incentive compatibility), (2) *individual rationality/selfishness*, and (3) *social welfare maximization (exclusion-compensation)*. We examine the PSP auction as the constraints are able to attain the desirable property of truthfulness through incentive compatibility, meaning that an user has more of an incentive to tell the truth. This is because in second-price markets, the winning bid does not pay the winning bid price, but the price from *next lowest bid*. The pricing mechanism also upholds the exclusion-compensation principle, or Pareto criterion, where any change to the system would make at least one user worse-off. We construct the model for a PSP data auction for mobile users participating in secondary mobile data exchange market.

Let the set of all wireless users to be labeled by the index set $\mathcal{I} = \{1, \cdots, I\}$. In our current formulation, we do not allow a seller to host multiple auctions, thus we may identify each local auction with the index of the seller $j \in \mathcal{I}$. The bid profiles of the users are given as, $s \equiv [s_i^j]$ where $(i, j) \in \mathcal{I} \times \mathcal{I}$. Now, this is a single bid, where we fill the space by submitting zero bids to all non-active users, meaning that if there is no interaction between two players i and j, then $(i, j) = 0$. One may think of it as an $\mathcal{I} \times \mathcal{I}$ matrix, with each element of the matrix representing a pair-interaction. However this matrix is just one projective representation of the space. A single snapshot of a static system, all quantities and prices are fixed may be represented by this matrix. Once users begin to bid, then we must consider all possible interactions, which is done by fixing one index in $\mathcal{I}$ at a time, allowing all other quantities to vary. So the strategy space in fact includes all the possibilities for an user in $\mathcal{I}$; another dimension to the problem is added with each possible variation. We call this space S, the (full) strategy space for buyer i as all possible bids at all auctions (where i's bid changes with respect to the variation of all other bids): $S_i = \Pi_{j \in \mathcal{I}} S_i^j$, and $S_{-i} = \Pi_{j \in \mathcal{I}} \big(\Pi_{k \neq i \in \mathcal{I}} S_k^j \big)$ as the associated opponent profiles, as in standard game-theoretic notation.

The grid(s) of bid profiles, s, represents the uncertain state of the distributed PSP auction mechanism in the secondary market, where we take uncertain to mean the statistical distribution of player types and corresponding actions. In general, we will not reference the full grid s. We will also use the context of the bid to indicate the user type. To further clarify our analysis, we adopt the following notational conventions: a seller's profile is denoted by $s^j = [s_i^j]_{i \in \mathcal{I}}$, and $s_i = [s_i^j]_{j \in \mathcal{I}}$ denotes a buyer's profile, where $s_{-i} \equiv [s_1^j, \cdots, s_{i-1}^j, s_{i+1}^j, \cdots, s_I^j]_{j \in \mathcal{I}}$ as the profile of user i's opponents. Furthermore, noting that this is a simplification for ease of notation, we let $D^j = \sum_{i \in \mathcal{I}} d_i^j$ be the total amount of data j has to sell, and $D_i = \sum_{j \in \mathcal{I}} d_i^j$ represent the total amount of data desired by buyer i.

We assume a public platform, published by the ISP, that allows sellers to advertise their auctions. Buyers may submit bids directly to sellers over the wireless network. We also assume that a buyer's budget is sufficient, as the alternative would be to pay a higher price to the ISP. We describe the rules as follows:

- The bid is represented by $s_i^j = (d_i^j, p_i^j)$, meaning i would like to buy from j a quantity d_i^j and is willing to pay a unit price p_i^j.
- The seller takes responsibility for notifying i of opponent bid profiles s_{-i}, and updates the bid profile when buyer i joins the auction.
- $s_i^j > 0$ represents a buyer-seller pair in s, with bid, $s_i^j =$

(d_i^j, p_i^j), where quantity $d_i^j \in d^j$ is an element of $[d_i^j]_{i\in\mathcal{I}}$, with reserve unit price $p_i^j \in p^j$, an element of $[p_i^j]_{i\in\mathcal{I}}$.

- If a buyer does not submit a bid to a seller, then this implies $s_i^j = 0$. A buyer that does not submit a bid will not receive opponent profiles from seller j.
- A user who does not submit a bid is holding to the previous bid, either zero or nonzero.

We emphasize that buyers are consistently referenced using the index i as a subscript, and sellers using the index j as a superscript, as in [8].

A. Market incentive.

We examine the role of buyers, who are able to directly influence global market dynamics, and assume that the sellers take a reactionary role. Each buyer i will have information from each seller j, as well as opponent profiles s_{-i}, from each auction in which it is participating. In the extreme case, where i submits bids to all auctions $j \in \mathcal{I}$, buyer i gains access all buyer profiles, $[s_1, \cdots, s_I]$. However, sellers can only gain information about the market by observing buyer behavior in their local auction. Buyers, on the other hand, can see all the sellers reserve prices, although they can only see their opponent bid profiles.

Define the set of sellers chosen by buyer $i \in \mathcal{I}$ as,

$$\mathcal{I}_i(n) = \underset{\mathcal{I}'\subset\mathcal{I}, |\mathcal{I}'|=n}{\arg\max} \sum_{j\in\mathcal{I}'} D^j,$$

and similarly, for a seller $j \in \mathcal{I}$, we define the set of buyers participating in auction j as,

$$\mathcal{I}^j(m) = \underset{\mathcal{I}'\subset\mathcal{I}, |\mathcal{I}'|=m}{\arg\max} \sum_{i\in\mathcal{I}'} p_i^j,$$

where $m, n \in \mathcal{I}$.

The PSP auction given in [2] is a set of simple and symmetric rules that closely follow market theory. We now formally define a PSP auction, which determines the actions buyers and sellers in the secondary market. We define an **opt-out function**, σ_i, associated with a buyer i as part of its type. Buyer i, when determining how to acquire a possible allocation a, will determine its bid quantities by,

$$\sigma_i(a) = [\sigma_i^j(a)]_{j\in\mathcal{I}}. \tag{1}$$

In a general sense, σ_i applies our user strategy to the PSP rules. The rules presented here incorporate of the opt-out function with the auction mechanism, and closely follows the work presented in [2]. The market price function, P_i, for a buyer in the secondary market can be described as follows:

$$\begin{aligned} P_i(z, s_{-i}) &= \sum_{j\in\mathcal{I}} \ \sigma_i^j \circ p_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j\in\mathcal{I}} \left(\inf \left\{ y \geq 0 : d_i^j(y, s_{-i}^j) \geq \sigma_i^j(z) \right\} \right), \end{aligned} \tag{2}$$

and is interpreted as the aggregate of minimum prices that buyer i bids in order to obtain data amount z given opponent profile s_{-i}. We note that in the following analysis the total minimum price for the buyer cannot be an aggregation of the *individual* prices of the buyers, as it is possible that the reserve prices of the sellers may vary. The maximum available quantity of data in auction j at unit price y given s_{-i}^j is:

$$d_i^j(y, s_{-i}^j) = \sigma_i^j \circ d_i^j(y, s_{-i}^j) = \left[D^j - \sum_{p_k^j > y} \sigma_k^j(a) \right]^+. \tag{3}$$

It follows from the upper-semicontinuity of D_i^j that for s_{-i}^j fixed, $\forall\ y, z \geq 0$,

$$\sigma_i^j(z) \leq \sigma_i^j \circ d_i^j(y, s_{-i}^j) \Leftrightarrow y \geq \sigma_i^j \circ p_i^j(z, s_{-i}^j). \tag{4}$$

The resulting data allocation rule is a function of the local market interactions between buyers and sellers over all local auctions, as is composed with i's opt-out value, so that for each $i \in \mathcal{I}$, the allocation from auction j is,

$$\begin{aligned} a_i^j(s) &= \sigma_i^j \circ a_i^j(s) \\ &= \min \left\{ \sigma_i^j(a), \frac{\sigma_i^j(a)}{\sum\limits_{p_k^j = p_i^j} \sigma_k^j(a)} d_i^j(p_i^j, s_{-i}^j) \right\}, \end{aligned} \tag{5}$$

noting that for the full allocation from all auctions we may simply aggregate over the seller pool.

Remark: The bid quantity $\sigma_i^j(a)$ and the allocation a_i^j are complementary. In fact, the buyer strategy is the first term in the minimum, the second term being owned by the seller.

Finally, we must have that the cost to the buyer adheres to the second price rule for each local auction, with total cost to buyer i,

$$c_i(s) = \sum_{j\in\mathcal{I}} p_i^j \left(a_i^j(0; s_{-i}^j) - a_i^j(s_i^j; s_{-i}^j) \right). \tag{6}$$

The cost to buyer i adds up the willingness of all buyers excluded by player i to pay for quantity a_i^j. i.e.

$$c_i^j(s) = \int_0^{a_i^j} p_i^j(z, s_{-i})\ dz.$$

This is the "social opportunity cost" of the PSP pricing rule.

IV. User Strategy

In any market, a buyer or seller would like to obtain the maximum amount of utility possible while staying within budget. The buyer's utility maximizes the amount of data allocated by the seller, while the seller's utility maximizes the cost of the data sold. Clearly, the cost is the product of the unit price and the desired allocation. We examine cases where the buyer has found an allocation that satisfies its demand AND price constraints, and define a strategic bid to a move to a better market position.

A. User valuation (strategic incentive).

We define each buyer as a user $i \in \mathcal{I}$ with quasi-linear utility function $u_i = [u_i^j]_{j\in\mathcal{I}}$. A buyers' utility function is of the form,

$$u_i = \theta_i \circ \sigma_i(a) - c_i, \tag{7}$$

where the composition of the elastic valuation function θ_i with σ_i distributes a buyers' valuation of the desired allocation a across local markets, submitting the strategic bid to multiple seller's auctions. The composition map represents the codomain of $\theta_i(\sigma_i)$, which is the same as the domain of $\sigma_i(a)$, and performs the function of restricting the buyer's domain to minimize $d^j p^j - c_i$, i.e., maximize u_i. Using this rule, we extend the PSP rules described in [8] in order to find equilibria in subsets of local data-exchange markets.

The sellers, $j \in \mathcal{I}$ are not associated with an opt-out function. We consider their valuation to be a functional extension of the buyers, where θ^j is constructed from buyer demand. We adopt the definition for an elastic valuation function as in [2], which allows for continuity of constraints imposed by the user strategies.

Definition IV.1. *(Elastic demand) [2] A real valued function,* $\theta(\cdot) : [0,\infty) \to [0,\infty)$*, is an* (elastic) valuation function *on* $[0, D]$ *if*

- $\theta(0) = 0$*,*
- θ *is differentiable,*
- $\theta' \geq 0$*, and* θ' *is non-increasing and continuous,*
- *There exists* $\gamma > 0$*, such that for all* $z \in [0, D]$*,* $\theta'(z) > 0$ *implies that for all* $\eta \in [0, z), \theta'(z) \leq \theta'(\eta) - \gamma(z - \eta)$*.*

We begin our analysis with buyer valuation θ_i. A buyers' valuation of an amount of data represents how much a buyer is willing to pay for a unit of data (bandwidth). This is equivalent to the bid price when given a fixed amount of data. The buyers' utility-maximizing bid (fixing the desired allocation $z \geq 0$) is a mapping to the lowest possible unit price,

$$f_i(z) \triangleq \inf\left\{y \geq 0 : \rho_i(y) \geq z,\ \forall\ j \in \mathcal{I}\right\}, \tag{8}$$

where $\rho_i(y)$ represents the demand function of buyer i at bid price $y \geq 0$. The market supply function is the extreme case of possible buyer demand, and acts as an "inverse" function of f_i. We have, for bid price $y \geq 0$, $\rho_i(y) = \sum_{j\in\mathcal{I}:p_i^j \geq y} D^j$. The utility-maximizing bid price is the lowest unit cost for the buyer to be able participate in all the auctions in $\mathcal{I}_i$, and corresponds to the maximum reserve price amongst the sellers.

From the perspective of the seller we have a more direct interpretation of valuation as revenue. The demand function of seller j at reserve price $y \geq 0$ is $\rho^j(y) = \sum_{i\in\mathcal{I}:p_i^j \geq y} \sigma_i^j(a)$. We define the "inverse" of the buyer demand function for seller j as potential revenue at unit price y, we have,

$$f^j(z) \triangleq \sup\left\{y \geq 0 : \rho^j(y) \geq z,\ \forall\ i \in \mathcal{I}\right\}. \tag{9}$$

Unsurprisingly, f^j maps quantity z to the highest possible unit data price.

We show that user valuation satisfies the conditions for an elastic demand function, based on (9). We first note that, in general (and so we omit the subscript/superscript notation), the valuation of data quantity $x \geq 0$ is given by, $\theta(x) = \int_0^x f(z)\ dz$. We propose the following Lemma,

Lemma IV.1. *(user valuation) For any buyer* $i \in \mathcal{I}$*, the valuation of a potential allocation* a *is,*

$$\theta_i \circ \sigma_i(a) = \sum_{j\in\mathcal{I}} \int_0^{\sigma_i^j(a)} f_i(z)\ dz. \tag{10}$$

Now, we may define seller j*'s valuation in terms of revenue,*

$$\theta^j = \sum_{i\in\mathcal{I}} \theta^j \circ \sigma_i^j(a) = \sum_{i\in\mathcal{I}} \int_0^{\sigma_i^j(a)} f^j(z)\ dz. \tag{11}$$

We have that θ_i *and* θ^j *are elastic valuation functions, with derivatives* ${\theta_i}'$ *and* ${\theta^j}'$ *satisfying the conditions of elastic demand.*

Proof. Let ξ be a unit of data from buyer bid quantity $\sigma_i^j(a)$. If ξ decreases by incremental amount x, then seller bid d_i^j must similarly decrease. The lost potential revenue for seller j is the price of the unit times the quantity decreased, by definition, $f^j(\xi)x$, and so, $\theta^j(\xi) - \theta^j(\xi - x) = f^j(\xi)x$, and (11) holds. As we may use the same argument for (10), as such, we will denote $f_i = f^j = f$ for the remainder of the proof. We observe that the function f is the first derivative of the valuation function with respect to quantity. Letting $\theta_i = \theta^j = \theta$, the existence of the derivative implies θ is continuous, and therefore, in this context, f represents the marginal valuation of the user, θ'. Also, clearly $\theta(0) = \theta(\sigma(0)) = 0$. Now, as we consider data to be an infinitely divisible resource, we have a continuous interval between allocations a and b, where $a \leq b$. Now, as θ is continuous, for some $c \in [a, b]$,

$$\theta'(c) = \lim_{x\to c} \frac{\theta(x) - \theta(c)}{x - c} = f(c),$$

and so $f = \theta'$ is continuous at $c \in [a, b]$, and so as $a \geq 0$, $\theta' \geq 0$. Finally, we have that concavity follows from the demand function. Then, as θ' is non-increasing, we may denote its derivative $\gamma \leq 0$, and taking the derivative of the Taylor approximation, we have, $\theta'(z) \leq \theta'(\eta) + \gamma(z - \eta)$. $\square$

Finally, it is worth mention that the analysis of the auction as a game only assumes some form of demand and supply, in order to derive properties. The mechanism itself does not require any knowledge of user demand or valuation.

B. User behavior.

Buyers and sellers are able to change their bid strategies asynchronously. A user's local strategy space is therefore non-deterministic as the preferences of users are subject to change. Although it is possible for a seller to fully satisfy a buyer i's demand, it is also reasonable to expect that a seller's overage data may not satisfy even a single buyer's demand. In this case, a buyer must split its bid among multiple sellers. The buyer strategy bids in auctions with the highest quantities first, a natural result of the demand curve.

The buyer strategy tends towards equal valuation of all local markets, and therefore similar prices. Buyer i's seller pool is determined by minimizing n, the smallest set of sellers that satisfy its demand D_i: $\min\{n \in \mathcal{I} \mid nD^n \geq D_i\}$. Similarly, seller j determines the minimal set of buyers that maximizes revenue and sells all of its data, D^j, i.e. $\min\{m \in \mathcal{I} \;\Big|\; \sum_{i \in \mathcal{I}^j(m)} d_i^j \geq D^j\}$. We use $j^* = n \leq I$ to represent the seller with the least amount of data $\in \mathcal{I}_i$, i.e. $D^{j^*} \leq D^j, \; \forall\, j \in \mathcal{I}^j$.

Define the composition,

$$\sigma_i^j \circ a = \sigma_i^j(a) = \frac{a_i^j}{|\mathcal{I}_i|}, \tag{12}$$

to be the buyer strategy with respect to quantity for all sellers $j \in \mathcal{I}_i$. We propose the following scheme:

Definition IV.2. *(Opt-out buyer strategy) Let $i \in \mathcal{I}$ be a buyer and fix all other buyers' bids s_{-i} at time $t > 0$, and let a be i's desired allocation. Define,*

$$\sigma_i^j(a) \triangleq \begin{cases} \sigma_i^{j^*}(a), & j \in \mathcal{I}^j, \\ 0, & j \ni \mathcal{I}^j. \end{cases} \tag{13}$$

and bid price $p_i^j = \theta_i'(\sigma_i^j(a))$.

Let the reserve price for seller j be,

$$p_*^j = p_{i^*}^j + \epsilon, \tag{14}$$

where i^* is the bidder with the highest "losing" bid price. A truthful bid implies that the new bid price differs from the last bid price by at least ϵ.

We will show that sellers are able to maximize revenue in a restricted subset of buyers in $\mathcal{I}$, and as such will attempt to facilitate a local market equilibrium for this subset. A local auction j converges when $s_i^{j(t+1)} = s_i^{j(t)} \; \forall\, i \in \mathcal{I}$, at which point the allocation is stable, the data is sold, and the auction ends. We propose a strategy to maximize (local) seller revenue.

Lemma IV.2. *(Localized seller strategy (i.e. progressive allocation)) For any seller j, fix all other bids $[s_i^k]_{i,k \neq j \in \mathcal{I}}$ at time $t > 0 \in \tau$. For each $t \in \tau$, let $\omega(t)$ be define the winner at time t, and perform the update,*

$$D^{j(t+1)} = D^{j(t)} - \sigma_{\omega(t)}^{j(t)}(a). \tag{15}$$

Allowing t to range over τ, we have that $D^j = 0$, and a local market equilibrium. We omit the proof, and provide a simple example.

C. A simple example.

We give a simple example of convergence to a local market equilibrium, where the buyers are assumed to respond according to (5).

Name	Bid total	Unit price
A	50	1
B	40	1.2
C	26	1.5
D	20	2
E	14	2.2

Let $s^{(1)} = [(65, \epsilon)]_{i \in \mathcal{I}}$ and $s^{(2)} = [(85, \epsilon)]_{i \in \mathcal{I}}$. The buyer bids are as follows:

$$\begin{aligned} s_A &= [(0,0),(50,1)], \\ s_B &= [(0,0),(40,1.2)], \\ s_C &= [(0,0),(26,1.5)], \\ s_D &= [(0,0),(20,2)], \\ s_E &= [(0,0),(14,2.2)]. \end{aligned}$$

Then at $t = 1, s^{(2)} = [(0,p^{(2)}),(20,p^{(2)}),(26,p^{(2)}),(20,p^{(2)}),(14,p^{(2)})]$, and so $(D^{(2)}, p^{(2)}) = (85, 1+\epsilon)$, The buyer response is,

$$\begin{aligned} s_A &= [(50,1),(0,0)], \\ s_B &= [(40,1.2),(0,0)], \\ s_C &= [(0,0),(26,p^{(2)})], \\ s_D &= [(0,0),(20,p^{(2)})], \\ s_E &= [(0,0),(14,p^{(2)})]. \end{aligned}$$

At $t = 2, (D^{(1)}, p^{(1)}) = (65, 1+\epsilon)$, with bid vector $s^{(1)} = [(25,p^{(1)}),(40,p^{(1)}),(0,0),(0,0),(0,0)]$. $(D^{(2)}, p^{(2)}) = (25, 1+\epsilon)$. Then,

$$\begin{aligned} s_A &= [(25,p^{(1)}),(25,p^{(2)})], \\ s_B &= [(40,p^{(1)}),(0,0)], \end{aligned}$$

where we have removed bids to indicate winner(s) with a tentative allocation. At $t = 3, (D^{(1)}, p^{(1)}) = (50, 1+\epsilon)$, with bid vector $s^{(1)} = [(25,p^{(1)}),(40,p^{(1)}),(0,0),(0,0),(0,0)].(D^{(2)},p^{(2)}) = (0, 1+\epsilon)$ and $s^{(2)} = [(25,p^{(1)}),(0,0),(26,p^{(2)}),(20,p^{(2)}),(14,p^{(2)})]$. Then,

$$s_A = [(25,p^{(1)}),(0,0))].$$

At $t = 4$ the auction ends.

1) Individual rationality/selfishness.: Value is modeled as a function of the entire marketplace: a buyer's valuation is aggregated over all the auctions, and the seller's valuation is aggregated over its own auction. We must ensure that a user's private action satisfies the conditions of a direct-revelation mechanism, as well as adheres to the collective goals. We show that, from (IV.2) and (IV.2), an individual user will contribute to local stability, given global market dynamics S.

We model the impact of the dynamics of S of the data-exchange market on a local auction j. As we have shown, the seller behavior is a reaction of buyer behavior, and have presented some rules. The market fluctuations from S give auctioneer j the chance to infer information about the global market. We demonstrate that the symmetry between buyer and seller behavior stretches across subsets of local auctions. Additionally, we identify a clear bound restricting the range of influence that local auctions have on each other. Consider a single iteration of the auction, where a seller updates bid vector s^j, and the buyers' response s_i, to comprise a single time step. We have the following Proposition,

Proposition IV.1. *(Valuation across local auctions) For any* $i, j \in \mathcal{I}$,

$$j \in \mathcal{I}_i \Leftrightarrow i \in \mathcal{I}^j. \tag{16}$$

Fix an auction $j \in \mathcal{I}$ *with duration* τ *and define the influence sets of users. The primary and secondary influencing sets are given as,*

$$\Lambda = \bigcup_{i \in \mathcal{I}^j} \mathcal{I}_i, \quad \textit{and} \quad \lambda = \bigcup_{i \in \mathcal{I}^j} \Big(\bigcup_{k \in \mathcal{I}_i} \mathcal{I}^k \Big). \tag{17}$$

Define $\Delta = \Lambda \cup \lambda$. *Fixing all other bids* $s_i^j \in \mathcal{I}$, *and time* $t > 0 \in \tau$, *we have that,*

$$\sum_{j \in \Lambda} \theta_i^j = \sum_{i \in \lambda} \theta_i^j. \tag{18}$$

Proof. As this is our main result, we provide an outline of the (exhaustive) proof, illustrating the most important case, when a market shifts affect auction j, and the direct influence of the shift on the connected subset of local markets.

A local auction $j \in \mathcal{I}$, is determined by the collection of buyer bid profiles. Using Proposition IV.2 and (16), we have that,

$$i \in \mathcal{I}^j \Leftrightarrow p_i^j > p_{i^*}^j, \tag{19}$$

where we define i^* as the losing buyer with the highest bid price in auction j. By (8) $p_i^j \geq p_{i^*}^j + \epsilon$, thus $p_i^j < p_{i^*}^j$ can only happen during a market shift. Consider $k \in \mathcal{I}^j$ at time t where, for example, some buyer(s) enter the auction, and so (19) implies that $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) > D^j$. Now, $p_i^j < p_{i^*}^j \Rightarrow k \ni \mathcal{I}^j$ and $s_k^j > 0$ will cause k to initiate a shift. By definition IV.2, k will set $s_k^j = 0$, and begin to add sellers to its pool. Suppose that at time t, j's market is at equilibrium. Unless k adds a seller with a higher reserve price within $|\mathcal{I}^j|$ time steps, by (15), the auction ends. We have that, $\forall\ i \in \mathcal{I}^j$, $\nexists\ s_i^j > 0$ where $i \ni \mathcal{I}^j$, and (16) holds.

Now, the subset $\mathcal{I}^j \subset \mathcal{I}$ determines j's reserve price $p_{i^*}^j$. We will assume the buyer submits a coordinated bid, using (5). The reserve price (14) of seller j is determined at each shift, and is the lowest price that j will accept to perform any allocation. Let p_*^j denote the reserve price of auction j and p_i^* denote the bid price of buyer i, i.e. $p_i^k = p_i^*,\ \forall\ k \in \mathcal{I}_i$. Using Proposition IV.2, for each $i \in \mathcal{I}^j$, we have from (8), (9), that $p_i^* \geq p_*^k,\ \forall\ k \in \mathcal{I}_i$. In the simplest case, consider a disjoint local market j, where $\forall\ i \in \mathcal{I}^j$, $s_i^k = 0,\ \forall\ k \neq j \in \mathcal{I}_i \Rightarrow \Lambda = \{j\}$ and $\lambda = \mathcal{I}^j$. Again using (8) and (9), it is clear that $\theta_i = \theta^j,\ \forall\ i \in \mathcal{I}^j$. In all other cases, the sellers $\in \Lambda$ are competing to sell their respective resources to buyers whose valuations are distributed across multiple auctions. The bid price of buyer $i \in \mathcal{I}^j$ is determined by, $p_i^* = \max_{k \in \mathcal{I}_i}(p_*^k)$. Λ is the set of sellers directly influencing the bids of buyers in auction j. Now, the reserve price for auction j is such that, $p_*^j \leq \min_{i \in \mathcal{I}^j}(p_i^*) - \epsilon$. From (17), Λ is defined by a seller $j \in \mathcal{I}$, where each user $k \in \lambda$ has some direct or indirect influence on j. Denote $\Delta^j = \Lambda^j \cup \lambda^j$.

Consider the set λ^j. For some buyer $i \in \mathcal{I}^j$, and then for some seller $k \in \mathcal{I}_i$, we have a buyer $l \in \mathcal{I}^k$. By (16), $i, l \in \mathcal{I}^k$, and so the reserve price $p_*^k \leq \min(p_l^*, p_i^*)$, and $k, j \in \mathcal{I}_i \Rightarrow p_i^* \geq \max(p_*^k, p_*^j)$. Suppose that $l \ni \mathcal{I}^j \Leftrightarrow j \ni \mathcal{I}_l$, so that $p_l^* < p_*^j$, and the valuation of buyer l does not impact auction j and vice versa, i.e. $\theta_l^j = 0$. Since $l \in \mathcal{I}^k$, $p_l^* \geq p_*^k \Rightarrow p_*^k < p_*^j$, and $i \in \mathcal{I}^j \Rightarrow p_i^* \geq p_*^j$. Therefore, we have that the ordering implied by (17) holds, and,

$$p_*^k \leq p_l^* < p_*^j \leq p_i^*, \tag{20}$$

for any buyer $l \in \lambda^j$ such that $l \ni \mathcal{I}^j$. We use a similar argument for a secondary user $q \in \mathcal{I}_l$.

Finally, consider the subset Λ^j; a shift occurs in 2 cases. (1) If $i \in \mathcal{I}^j$ decreases its bid quantity so that $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) < D^j$, and (2) if buyer i^*, defined in Proposition IV.2, increases its valuation so that $p_{i^*}^j < p_*^j$. Fixing all other bids, a decrease in q's demand will directly impact buyer i. If at the end of the bid iteration, we still have that i is the buyer with the lowest bid price, then (9) holds and j's valuation does not change. Otherwise a new i^* will be chosen upon recomputing $\mathcal{I}^j$, as in Lemma IV.2, and the market will attempt to regain equilibrium. We determine the influence of Δ^{k^*} on Δ^j by (19).

In each case we have that (8) and (9) hold for some fixed time t, and so, $\forall\ i \in \mathcal{I}^j$, any bid outside of our construction has a zero valuation, with respect to buyers $\in \lambda$ and sellers $\in \Lambda$, and therefore cannot cause shifts to occur except through a shared buyer, e.g. some $l \in \mathcal{I}^k$. Thus, in all cases, (8) and (9) hold. Fixing all bids in any auction where $q \ni \Lambda^j$, $\forall\ i \in \mathcal{I}^j, \forall\ k \in \mathcal{I}_i, \forall\ l \in \mathcal{I}^k$,

$$\int_0^{\sigma_i^k(a)} f_i(z)\ dz = \int_0^{\sigma_i^k(a)} f^k(z)\ dz, \tag{21}$$

and

$$\int_0^{\sigma_l^k(a)} f^k(z)\ dz = \int_0^{\sigma_l^k(a)} f_l(z)\ dz. \tag{22}$$

Thus, with a slight abuse of notation for clarity,

$$\sum_{\lambda} \int_0^{\sigma(a)} f^{\Lambda}(z)\ dz = \sum_{\Lambda} \int_0^{\sigma(a)} f_{\lambda}(z)\ dz, \tag{23}$$

where the result follows by construction, and the continuity of θ'. $\square$

2) Truthfulness (incentive compatibility).: We will prove that the dominant strategy for buyers is to submit coordinated bids, where all bids the buyer submits are equal. Our motivation for coordinated bids comes from the idea of potential games. In potential games, the incentive of all users to change strategy can be expressed as a single global function. The necessary condition of an ϵ-best reply is that the new bid price must differ from the last by at least ϵ. Thus, our strategic bid is an ϵ-best response. Now, an ϵ-best reply for user i is $p_i^* = \theta_i'(\sigma_i(a)) + \epsilon$, for a given opponent profile s_{-i}, and for each $j \in \mathcal{I}_i$. Now, as ϵ is the bid fee, we have that p_i^j is equal to the marginal valuation of player i in auction j, and so incentive compatibility holds.

3) Social welfare maximization (exclusion-compensation).: We define an optimal state of social welfare to be when valuations are equal across a subset of local auctions. Then, $\Delta \subset \mathcal{I}$ is the subset of users where social welfare is achieved. We finally have:

Corollary IV.1. *(Δ-Pareto efficiency) The subset $\Delta \subset \mathcal{I}$ is Pareto efficient, in that no user can make a strategic move without making any other user worse off.*

Proof. Define $s_* = (z_*, \theta_*'(z_*)$ as the set of truthful ϵ-best replies for user i given opponent bid profile S_{-i}, where $\forall j \in \mathcal{I}_i$, $s_*^j = s_*$. Since θ_i' is continuous, as was shown in Lemma IV.1, and as $s|_\Delta = \{[s_i^j] \in \lambda^j \times \Lambda^j\}$ is continuous in s on $S_k = \Pi_{k \in \lambda^j} S_k^j$, then given that $s_* = s^* = (f^*(p^*), p^*) = (z^*, \theta'(z^*))$, we have that s^* is truthful. The result now follows directly from the result of Proposition IV.1. $\square$

V. Conclusion and Future Work

We take these results as evidence of (at least one) fixed point, and conjecture that an optimal solution exists, where all users will receive the desired amount of data (negative or positive), at a fair price.

The PSP auction is a natural data-pricing scheme for consumers accessing a data-exchange market in their wireless network, and that the desired properties of (1) *truthfulness*, (2) *individual rationality/ selfishness*, and (3) *social welfare maximization* are met. We conclude that there is a need for better management of data on the consumer level; an advanced implementation such as the PSP auction presented here would ensure that the consumers in any such exchange market benefit from their participation. It is clear that there is profit to be made by supplying data to the data-driven consumer. However, customer care is necessary to hold the "lifetime consumer". Consumers, when allowed to manage their own overage data, are able to do so fairly and efficiently. It is not unreasonable to allow them to manage their own data; this benefits all wireless users.

Mathematically, we have shown that if truthfulness holds locally for both buyers and sellers, i.e. $p_i = {\theta_i}'$, $\forall\ j \in \mathcal{I}_i$ and $p^j = {\theta^j}'$, $\forall\ i \in \mathcal{I}^j$, then, in the absence of market shifts, there exists an ϵ-Nash equilibrium extending over a subset of connected local markets. Observing the symmetric, natural topology of the strategy space, we conjecture that a unique subspace limit exists for connected Δ. A study of this space and the design of the necessary framework is the direction of our future work.

In future work, we intend to show that $s|_\Delta$ represents a continuous mapping $[0, \sum_{k \in \lambda^j} D^k]_{i \in \Lambda^j}$ onto itself, and show that the continuous mapping of the convex compact set s_* into itself (s^*) has at least one fixed point, i.e., $\exists$ some $k \neq i$ such that $z^* = \sigma^*(z) \in [0, D_k]_{i \in \Lambda^j}$. We want to show that the symmetry built into strategy space provides built-in conditions for convergence and stability, indicating a network Nash equilibrium (NE). Wireless users are modeled as a distribution of buyers and sellers with normal incentives.

Finally, as a result of user behavior and subsequent strategies, we determine that the data-exchange market behaves in a predictable way. However, each auction may be played on the same or on different scales in valuation, time, and quantity; therefore the rate at which market fluctuations occur is impossible to predict. Nonetheless, we have shown that our bidding strategy results in (at least one), Nash equilibrium, where again the reserve prices are fixed by the seller at bid time.

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