PAPER

On the crossing number of a torus network

Antoine BOSSARD[†], *Nonmember*, **Keiichi KANEKO**[††], *Member*, *and* **Frederick C. HARRIS, JR.**[†††], *Nonmember*

SUMMARY Reducing the number of link crossings in a network drawn on the plane such as a wiring board is a well-known problem, and especially the calculation of the minimum number of such crossings: this is the crossing number problem. It has been shown that finding a general solution to the crossing number problem is NP-hard. So, this problem is addressed for particular classes of graphs and this is also our approach in this paper. More precisely, we focus hereinafter on the torus topology. First, we discuss an upper bound on $cr(T(2,k))$ the number of crossings in a 2-dimensional k-ary torus $T(2,k)$ where $k \geq 2$: the result $cr(T(2,k)) \leq k(k-2)$ and the given constructive proof lay foundations for the rest of the paper. Second, we extend this discussion to derive an upper bound on the crossing number of a 3-dimensional k-ary torus: $cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2 - 2\lceil k/2 \rceil \lfloor k/2 \rfloor (k - (k \bmod 2))$ is obtained. Third, an upper bound on the crossing number of an n-dimensional k-ary torus is derived from the previously established results, with the order of this upper bound additionally established for more clarity: $cr(T(n,k))$ is $O(n^2 k^{2n-2})$ when $n \geq k$ and $O(nk^{2n-1})$ otherwise.
key words: *interconnect, network, intersection, graph, planar*

1. Introduction

The minimum number of link crossings when drawing a graph on the plane is called the crossing number of that graph (a formal definition of the crossing number of a graph will be given in the next section). There are several important applications for this well-studied graph drawing problem, and in multiple fields. For example, it is critical for circuit design (as that for VLSI) to minimise, and possibly reduce to zero, the number of link crossings so that the circuit can be easily realised (printed) on a board [1]–[3]. Graph visualisation is another application example for the crossing number problem [4]. Solving this problem is notoriously difficult: it has been proved that the crossing number problem is NP-hard [5]. This problem has been discussed in the general case (i.e., for any graph) in [6], [7], and relatively more recently in [8], to only cite a few.

Because of the prohibitive complexity when considering the general case, this problem has been instead addressed for special classes of graphs. For example, [9] discussed the crossing number of hypercubes, with new findings described by [10]. Precisely, it has been shown in [10] that $4^n \cdot 5/32 - \lfloor (n^2+1)/2 \rfloor 2^{n-2}$ is an upper bound on the crossing number of an n-dimensional hypercube. Complete graphs [11] and stars [12] are additional examples of specific graph classes for which this problem has been discussed.

Another class of graphs is considered in this paper: torus networks. An extended abstract of this paper has been published by the authors [13].

The rest of this paper is organised as follows. Several definitions, notations and previously established results with respect to the addressed problem are recalled in Section 2. Then, the case of a 3-dimensional k-ary torus is discussed in depth in Section 3, with an upper bound on the crossing number induced. Finally, an upper bound on the crossing number of an n-dimensional k-ary torus is derived in Section 4 from the obtained results, and the corresponding order is calculated. This paper is concluded in Section 5.

2. Preliminaries

In this section, several definitions and notations are recalled. First, regarding graph theory notations, it is recalled that a graph G is made of vertices (a.k.a. nodes) and edges. The number of vertices of G is denoted by $|G|$, and the number of edges of G by $||G||$.

Definition 1. [14] *An n-dimensional mesh is an undirected graph that has k_i nodes on the i-th dimension ($k_i \geq 2$, $1 \leq i \leq n$), inducing $\prod_{i=1}^{n} k_i$ nodes in total. The address of a node u has n coordinates $(u_1, u_2, \ldots, u_n)$ with $0 \leq u_i \leq k_i - 1$ ($1 \leq i \leq n$). Two nodes u, v are adjacent if and only if $\exists j$ ($1 \leq j \leq n$) such that $\forall i$ ($1 \leq i \leq n$, $i \neq j$) $u_i = v_i$ and either $u_j = v_j + 1$ or $u_j = v_j - 1$.*

Definition 2. [14] *An n-dimensional k-ary torus $T(n,k)$, $n \geq 1$, $k \geq 1$, is an undirected graph whose k^n nodes are the n-vectors induced by the set $\{0, 1, \ldots, k-1\}^n$. Two vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ of a $T(n,k)$ are adjacent if and only if $\exists j$ ($1 \leq j \leq n$) such that $\forall i$ ($1 \leq i \leq n$, $i \neq j$) $u_i = v_i$ and $u_j = v_j \pm 1 \pmod{k}$.*

Two sample tori, precisely a $T(2,4)$ and a $T(3,3)$, are shown in Figure 1. For the sake of figure clarity, the node addresses are mentioned only for $T(2,4)$.

Then, we recall several definitions, notations and results with respect to the crossing number problem [15]. A *point* is a geometrical coordinate, and is not to be confused with a node (a.k.a. vertex) of a graph. Yet, one node of a graph induces one point. For a graph G, a *drawing* of G is the representation of G (i.e., its nodes and edges) on a surface, typically a plane, such as a sheet of paper. Such a drawing

[†]The author is with the Graduate School of Science, Kanagawa University
[††]The author is with the Graduate School of Engineering, Tokyo University of Agriculture and Technology
[†††]The author is with the Department of Computer Science and Engineering, University of Nevada, Reno
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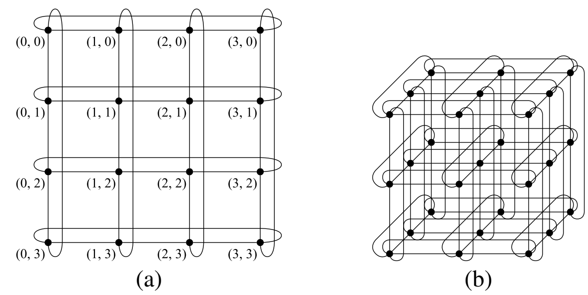


Fig. 1 (a) A 2-dimensional 4-ary torus $T(2, 4)$. (b) A 3-dimensional 3-ary torus $T(3, 3)$.

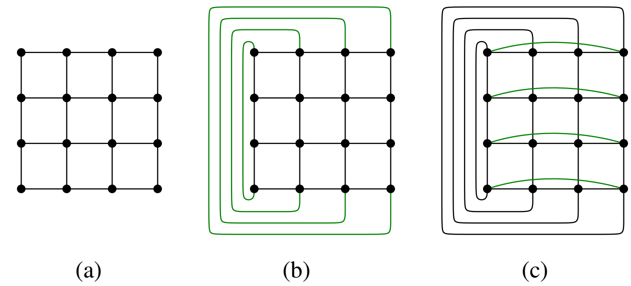


Fig. 2 Proposed construction process for a $T(2, k)$ in three steps: (a) to (c). Here, $k = 4$.

links the points corresponding to the nodes of G with arcs, each arc corresponding to one edge of G. In other words, any two points linked by such an arc are induced by two adjacent nodes of G. For a graph G, an *embedding* of G on a surface S corresponds to a drawing of G onto S where any two arcs of the drawing are allowed to intersect only at the point they both connect. In the case where the considered surface S is a plane, the graph that corresponds to such an embedding is said to be *planar*. More precisely, a graph that *can* be embedded on a plane is said to be *planar* and a planar graph embedded in the plane is called a *plane graph*.

A *region* of a plane graph G is a maximal portion of the plane inside which any two points can be joined by a curve in such a way that each point of this curve is not a vertex of G and is not included in the curve induced by an edge of G. The *regions* of a graph drawing are thus the complement of the union of the arcs of the drawing, that is, the complement of the set of all the points that make the arcs. For a region R, R is a *2-cell* if any closed curve contained by R can be progressively contracted to one point. For example, a region that contains a "hole" (the surface that includes such a region thus has a "hole", that is, the surface genus is at least one) is not a 2-cell, and, similarly, two concentric circles induce three regions, one of which is not a 2-cell. For an embedding E, if all the regions of E are 2-cells, E is said to be a *2-cell embedding*.

Now that a 2-cell embedding has been defined, the Euler formula can be recalled in the following theorem.

Theorem 1. *Euler's formula. For G a connected graph of n vertices, m edges and with a 2-cell embedding of r regions, we have $n - m + r = 2$.*

Finally, for a graph drawing on a plane, a *crossing* corresponds to a point that is included by exactly two distinct arcs while not being any one of the four endpoints of the two arcs. As such, a crossing is induced by one pair of distinct arcs. Furthermore, it should be noted that for any two distinct arc pairs that each induce a crossing, a total of two crossings are induced, independently of the respective points of the two crossings, that is, even if these two crossings correspond to the same point on the plane.

Definition 3. *For a graph G, the crossing number of G, denoted by $cr(G)$, is the minimum number of crossings among the drawings of G on a plane.*

From this definition, we directly have that a graph G satisfies $cr(G) = 0$ if and only if G is planar.

Before going further, the crossing number problem for particular subclasses of the torus class of graphs is discussed here so that these subclasses can be safely ignored in the rest of this paper (except a lemma, theorems and a corollary which remain self-contained). Consider a torus $T(n, k)$.

Case $n = 1$
The corresponding torus $T(1, k)$ is isomorphic to a ring of k nodes and thus planar.

Case $k = 1$
The corresponding torus $T(n, 1)$ consists of one single node and is thus planar.

Case $k = 2$
The corresponding torus $T(n, 2)$ is isomorphic to an n-dimensional hypercube. As recalled in introduction, it has been shown that $4^n \cdot 5/32 - \lfloor (n^2 + 1)/2 \rfloor 2^{n-2}$ is an upper bound on the crossing number of an n-dimensional hypercube [10].

Hence, we can assume hereinafter that $n \geq 2$ and $k \geq 3$.

Next, we discuss the case of a $T(2, k)$. This two-dimensional case (i.e., $n = 2$), often referred to as the Cartesian product of cycles $C_k \times C_k$, has been largely discussed in the literature; for instance, several upper bounds on $cr(T(2, k))$ are given in [16] and an approximation algorithm in [17]. We give with Lemma 1 a constructive proof from which an upper bound on $cr(T(2, k))$ can be derived. An illustration of the torus drawing process described in this lemma is given in Figure 2. Lemma 1 recalls a well-established result (see for instance [18], [19]) but gives a constructive proof that is essential for the understanding of the rest of the paper.

Lemma 1. *The crossing number of a $T(2, k)$ with $k \geq 3$ satisfies $cr(T(2, k)) \leq k(k - 2)$.*

Proof. First, a 2-dimensional k-ary mesh is considered. It is recalled that a mesh is a planar graph that has $(k - 1)^2 + 1$ regions, including the outer, unbounded one. See Figure 2a.

Second, considering one of the two torus dimensions, the corresponding k wrap-around edges are drawn in a way that no crossing emerges. See Figure 2b. In total, k new

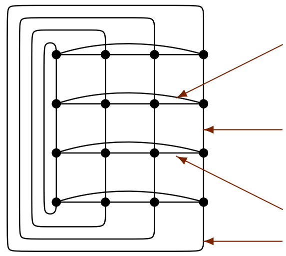


Fig. 3 Edge classification in a subtorus $T(2,k)$ (here, $k = 4$).

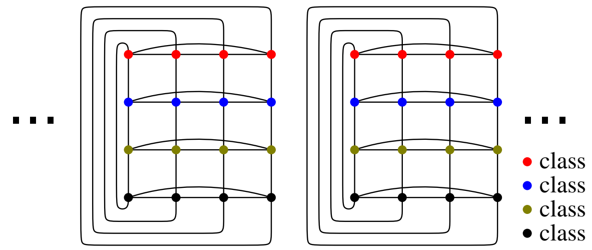


Fig. 4 The classes of the nodes of subtori.

regions are induced by these new edges. So, the total number of regions is now equal to $(k-1)^2 + k + 1$. It can be confirmed with Theorem 1 that the resulting graph remains planar: $k^2 - [2k(k-1)+k] + [(k-1)^2 + k + 1] = 2$.

Third, the k wrap-around edges that correspond to the other torus dimension are drawn. Unlike previously, each such new edge induces crossings, precisely at least $k-2$ crossings per such new edge. See Figure 2c. Hence, the minimum number of crossings when drawing a $T(2,k)$ according to this method is $k(k-2)$. □

It can be noticed that the optimal drawing of $T(2,k)$ is not known but the entire results of this paper are based on the drawing shown in the proof of Lemma 1.

3. An upper bound on the crossing number of a $T(3,k)$

We give in this section a constructive proof regarding an upper bound on $cr(T(3,k))$ the crossing number of a 3-dimensional torus. This proof is given in three successive parts.

For the sake of clarity, the edges of a subtorus $T(2,k)$ are categorised into four classes which are given the following names: wrap-around horizontal (i.e., edges that connect any $u = (0,i)$ and $v = (k-1,i)$ with $0 \le i \le k-1$), wrap-around vertical (i.e., edges that connect any $u = (i,0)$ and $v = (i,k-1)$ with $0 \le i \le k-1$), internal horizontal (i.e., edges that connect any $u = (i,j)$ and $v = (i+1,j)$ with $0 \le i \le k-2$, $0 \le j \le k-1$) and internal vertical (i.e., edges that connect any $u = (i,j)$ and $v = (i,j+1)$ with $0 \le i \le k-1$, $0 \le j \le k-2$). This edge classification is further detailed in Figure 3.

3.1 Connecting subtori

Here is described the first step of the torus construction. The approach followed is to rely on the recursive property of tori: a torus $T(3,k)$ is made of k subtori $T(2,k)$. More concretely, we connect 2-dimensional subtori $T(2,k)$ each other according to the topology of a 3-dimensional torus $T(3,k)$, with each of the k subtori $T(2,k)$ being drawn as detailed in Section 2. In general, in a $T(n,k)$, an edge that connects two nodes of distinct $T(n-1,k)$ subtori is called an *external edge*.

Consequently, the number of crossings induced by such a drawing method is equal to:

$$cr(T(3,k)) \le k \cdot cr(T(2,k)) + \alpha \qquad (1)$$

with α denoting the number of crossings that are induced by the external edges (i.e., crossings that involve at least one external edge).

Here, one should note that even if the $T(2,k)$ subtori are drawn seemingly optimally regarding the crossing number as described in Section 2, it does not ensure that a $T(3,k)$ with the least number of crossings can be obtained by connection of such optimal drawings of $T(2,k)$ subtori. Formally, while (1) holds as explained above, the equality $cr(T(3,k)) = k \cdot cr(T(2,k)) + \alpha$ remains to be shown – or refuted.

We start drawing $T(3,k)$ by k copies of the drawing of $T(2,k)$ lined up in a horizontal row, where the drawing of $T(2,k)$ is given in the proof of Lemma 1 (see Figure 4). It can thus be assumed that a subtorus is on the left (resp. right) of another, at the exception of the leftmost subtorus which is on the right of none, and of the rightmost subtorus which is on the left of none.

Definition 4. *For any two consecutive subtori T_1, T_2 with, say, T_1 on the left of T_2, the external edges that connect nodes of T_1 and T_2 are called* IN *external edges for T_2 and* OUT *external edges for T_1.*

Note that for T_l, T_r the leftmost and rightmost subtori, respectively, the external edges that connect nodes of T_l and T_r are similarly called IN external edges for T_l and OUT external edges for T_r.

Furthermore, each vertex u of such a $T(2,k)$ subtorus drawing is classified according to its vertical position, named *class* and denoted by class(u), as follows. Given a vertex $u = (u_1, u_2) \in T(2,k)$, we have class$(u) = u_2$, with thus $0 \le$ class$(u) \le k-1$ and the class 0 designating by convention the topmost vertices of a subtorus (i.e., the vertices of the topmost "row"). The class is then incremented for each vertex row, top to bottom. See Figure 4.

Next, the external edges are drawn depending on the class of subtorus nodes, with three cases – class sets – distinguished as follows. Let u be a node of $T(2,k)$.

Case class$(u) = 0$

The IN external edges end at such nodes u from above in a way that these edges do not cross each other and avoid wrap-around vertical edges as much as possible. The OUT external edges end at such nodes u from below in a way that these edges do not cross each other on their way to the nodes of the next subtorus.

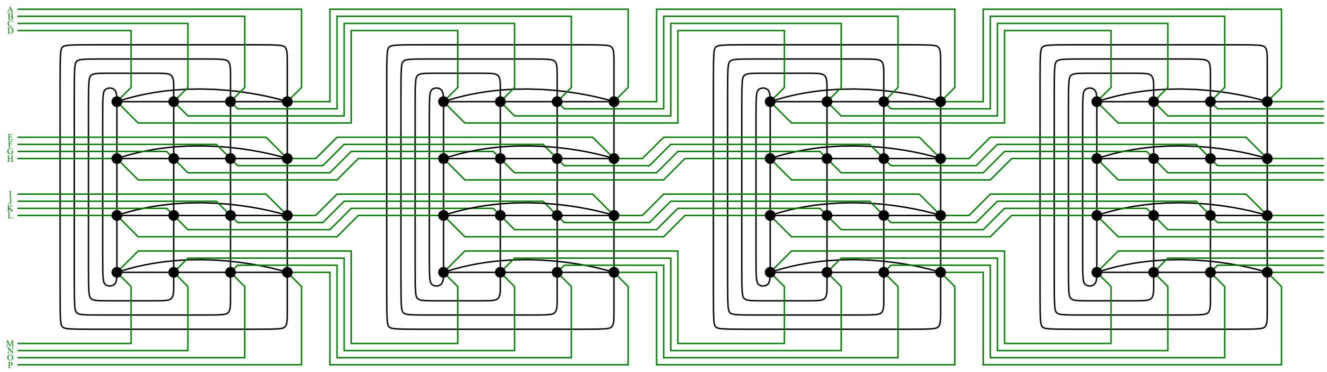


Fig. 5 Subtorus ground-laying connection. Subtori are connected each other with external edges (in green).

Case $1 \leq \text{class}(u) \leq k-2$
The IN external edges end at such nodes u from above in a way that these edges do not cross each other. The OUT external edges end at such nodes u from below in a way that these edges do not cross each other on their way to the nodes of the next subtorus.

Case $\text{class}(u) = k-1$
The IN external edges end at such nodes u from below in a way that these edges do not cross each other and avoid wrap-around vertical edges as much as possible. The OUT external edges end at such nodes u from above in a way that these edges do not cross each other on their way to the nodes of the next subtorus.

The described drawing method of a $T(3,k)$ is illustrated in Figure 5, where the external edges are coloured in green. In addition, in this figure the edges from A to P are cut solely for layout reasons; no additional crossing is induced by those given that the A–P labels are assigned in the same order on both side of the drawing and can thus be connected by uncrossed continuous lines for instance above the drawing.

Therefore, given this subtorus interconnection method, it is guaranteed that there does not exist a crossing between any two external edges. That is, crossings involve either two subtorus edges or one subtorus edge and one external edge.

Next, we count the number of crossings that are induced by this $T(3,k)$ drawing method. Since the number of crossings induced by the drawing of one subtorus $T(2,k)$ as detailed in Section 2 has already been established, the remaining task is to calculate α, the number of crossings that are induced by external edges (refer to (1)). The value of α will be established by calculating first β, the number of such crossings at one subtorus $T(2,k)$ (i.e., the number of crossings between an external edge and an edge of the subtorus), second multiplying β by k, the number of subtori. Hence, we have $\alpha = k\beta$.

The number β is established by distinguishing the same three class sets as previously. For the sake of clarity, we define $\#C$ as the total number of crossings induced at one subtorus by the external edges that end at the subtorus nodes u with $\text{class}(u) \in C$ (the set C is thus a class set). Hence, we have:

$$\beta = \#\{0\} + \#\{1,2,\ldots,k-2\} + \#\{k-1\}$$

The details of the calculation of $\#C$ for the three class sets $C \in \{\{0\},\{1,2,\ldots,k-2\},\{k-1\}\}$ is next given. The number of crossings is counted separately for IN and OUT external edges for each class set. The summations from 1 to k represent the iteration of the horizontal positions of subtorus nodes, the vertical positions being already treated by class set distinction.

Calculation of $\#\{0\}$
The number of crossings induced by the IN external edges is as follows:

$$\underbrace{(k-2)}_{\text{wrap-around horizontal}} + \underbrace{\sum_{i=1}^{k}(k-i)}_{\text{wrap-around vertical}} = \frac{k^2+k}{2} - 2$$

The number of crossings induced by the OUT external edges is as follows:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}} = \frac{k^2-k}{2}$$

Therefore, we have:

$$\#\{0\} = \frac{k^2+k}{2} - 2 + \frac{k^2-k}{2} = k^2 - 2$$

Calculation of $\#\{1,2,\ldots,k-2\}$
First, for one such class.
The number of crossings induced by the IN external edges is as follows:

$$\underbrace{k^2}_{\text{wrap-around vertical}} + \underbrace{(k-2)}_{\text{wrap-around horizontal}} + \underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}} = \frac{3k^2+k}{2} - 2$$

The number of crossings induced by the OUT external edges is as follows:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}} = \frac{k^2-k}{2}$$

Therefore, the number of crossings for each class is:

$$\frac{3k^2+k}{2} - 2 + \frac{k^2-k}{2} = 2k^2 - 2$$

and, considering all these classes, we have:

$$\#\{1,2,\ldots,k-2\} = \underbrace{(k-2)}_{\substack{\text{for each class}\\ \text{in } \{1,2,\ldots,k-2\}}} \; (2k^2-2)$$

$$= 2k^3 - 4k^2 - 2k + 4$$

Calculation of $\#\{k-1\}$

By symmetry of $\#\{0\}$, we directly have: $\#\{k-1\} = k^2 - 2$

We can thus derive the value of β:

$$\beta = 2k^3 - 2k^2 - 2k$$

and subsequently that of α:

$$\alpha = 2k^4 - 2k^3 - 2k^2 \quad (2)$$

This initial discussion regarding an upper bound on $cr(T(3,k))$ the crossing number of a 3-dimensional torus is summarised by Lemma 2.

Lemma 2. *The crossing number of a* $T(3,k)$ *satisfies the following relation:*

$$cr(T(3,k)) \leq \max\{0, 2k^4 - k^3 - 4k^2\}$$

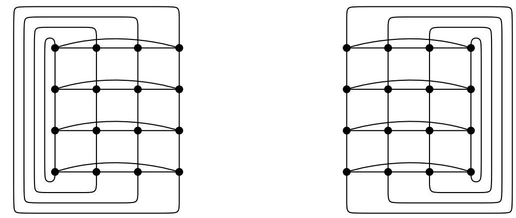
Proof. As explained, the case $k = 1$ induces a planar graph, and the case $k = 2$ induces the relation $cr(T(3,2)) = 0$ due to [10], and thus a planar graph as well. Regarding the case $k \geq 3$, this can be easily derived from (1) and (2). □

3.2 Flipping subtori

In this section, we describe a modification to the drawing method of Section 3.1. While this modification does not change the number of crossings as detailed below, and thus the upper bound on $cr(T(3,k))$, it is nonetheless important in that it is at the core of the improvement presented in Section 3.3.

First, we show that horizontally flipping (i.e., a 180° rotation around the vertical axis) a subtorus $T(2,k)$ does not affect the number of crossings of $T(3,k)$. The refined drawing method is detailed below. The same three class sets are distinguished, but in addition, subtori are categorised either as "flipped" or "non-flipped". A flipped subtorus $T(2,4)$ is shown in Figure 6b and a non-flipped subtorus $T(2,4)$ in Figure 6a.

First, in the case of a non-flipped subtorus, the drawing of external edges at this subtorus is the same as that of Section 3.1. In the case of a flipped subtorus, the drawing of external edges is as follows. Let u be a node of $T(2,k)$.



(a) (b)

Fig. 6 (a) The drawing of a non-flipped subtorus $T(2,4)$. (b) The drawing of a flipped subtorus $T(2,4)$.

Case $\text{class}(u) = 0$

The IN external edges end at such nodes u from below in a way that these edges do not cross each other. The OUT external edges end at such nodes u from above in a way that these edges do not cross each other and avoid wrap-around vertical edges as much as possible.

Case $1 \leq \text{class}(u) \leq k-2$

The IN external edges end at such nodes u from below in a way that these edges do not cross each other. The OUT external edges end at such nodes u from above in a way that these edges do not cross each other on their way to the nodes of the next subtorus.

Case $\text{class}(u) = k-1$

The IN external edges end at such nodes u from above in a way that these edges do not cross each other. The OUT external edges end at such nodes u from below in a way that these edges do not cross each other and avoid wrap-around vertical edges as much as possible.

This refined drawing method of a $T(3,k)$ is illustrated in Figure 7. The remarks made in Section 3.1 regarding the edges A–P and the fact that there is no crossing between any two external edges still hold.

Next, the number of crossings induced by this refined drawing method is calculated. We proceed as previously, but this time distinguishing flipped and non-flipped subtori. So, first, while (1) still holds, α is this time defined as follows:

$$\alpha = k_f \beta_f + k_n \beta_n$$

with $k_f + k_n = k$ and where k_f, k_n are the number of flipped and non-flipped subtori, respectively, and β_f, β_n are the number of crossings that are induced by external edges inside a flipped and non-flipped subtorus, respectively.

The value of β_n is obviously that of β calculated previously. We establish β_f below, that is, we calculate the total number of crossings induced at one flipped subtorus t by the external edges that end at the nodes $u \in t$ with $\text{class}(u) \in C$ for the three distinguished class sets $C \in \{\{0\}, \{1,2,\ldots,k-2\}, \{k-1\}\}$. Because the flipped subtorus is the mirror image of a non-flipped subtorus, calculation details are abbreviated since matching those of Section 3.1.

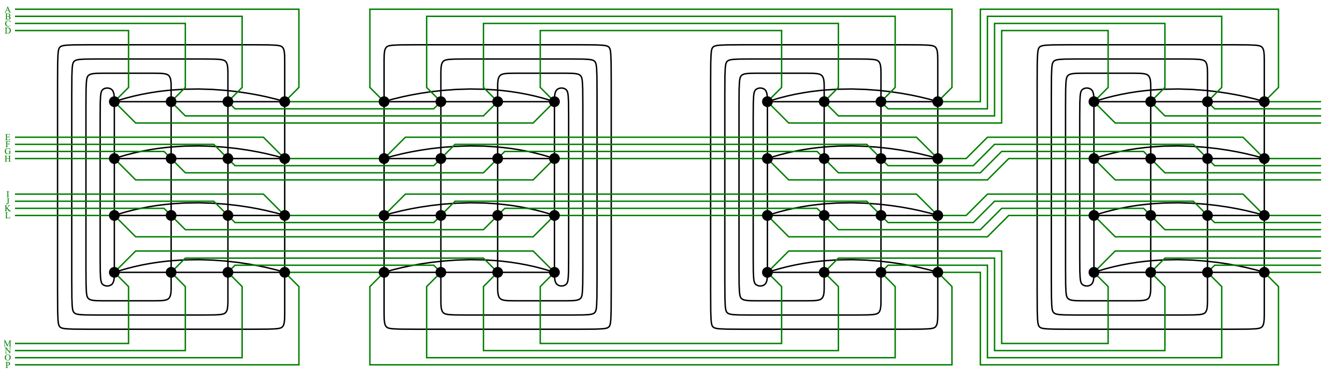


Fig. 7 Flipping subtori as described does not impact the number of crossings. Here, one subtorus is flipped.

As a result, we have that $\beta = \beta_f = \beta_n$. Subsequently, even if α was refined in this drawing method to $\alpha = k_f\beta_f + k_n\beta_n$, the relation $\alpha = k\beta$ still holds. Therefore, the same upper bound $cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2$ is induced by this refined drawing method of a $T(3,k)$.

In the remaining of this section, we describe the subtorus flipping scheme that will be used in Section 3.3.

Definition 5. *A subtorus pair consists of the drawing of two adjacent subtori $T(2,k)$ such that one is flipped and the other is not.*

We show in Section 3.3 that a few external edges per subtorus pair can be drawn differently to induce a smaller number of crossings. Therefore, we show here how to maximise the number of subtorus pairs. The flipping scheme detailed in Figure 8 is used; the cases k even and k odd are distinguished. Precisely, subtori are alternately flipped left to right, starting from a non-flipped subtorus. When k is even, each of all subtori is member of a subtorus pair, and pairs are formed as shown in Figure 8. When k is odd, one unique subtorus remains unpaired: it does not matter whether it is flipped; pairs are formed as shown in Figure 8. This flipping scheme is optimal in that it maximises the number of paired subtori. In Figure 8, the unpaired subtorus is displayed without loss of generality as the leftmost subtorus, and non-flipped.

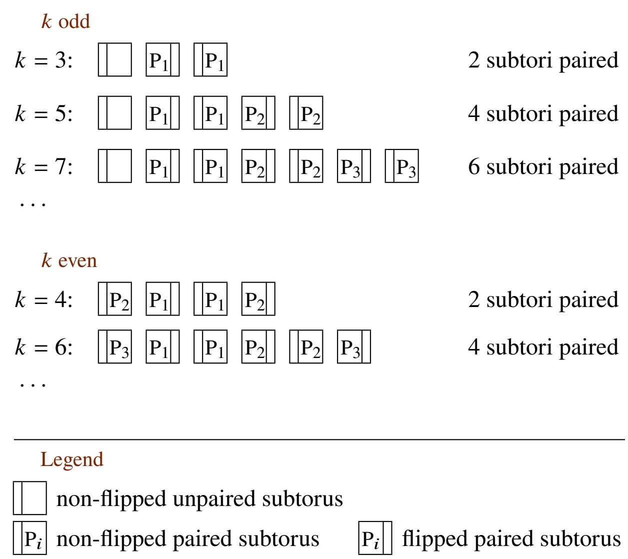


Fig. 8 Subtorus flipping scheme. Subtorus pairs are induced.

3.3 Improvement

In this section, we show that by maximising the number of subtorus pairs as described previously, a few external edges per subtorus pair can be drawn differently, inducing a smaller number of crossings.

Definition 6. *For a subtorus pair T_f, T_n with T_f (resp. T_n) flipped (resp. non-flipped), an external edge between $u = (u_1, u_2, u_3) \in T_f$ and $v \in T_n$ with $0 \leq u_1 < \lfloor k/2 \rfloor$ and* class$(u) \in \{1, k-2\}$ *as drawn in Figure 9 is called a modified external edge.*

First, we consider the modified external edges for each subtorus pair T_f, T_n with T_f (resp. T_n) flipped (resp. non-flipped). Modified external edges thus replace external edges which originally went through the two paired subtori but now go around them from above and below. Due to the modification, the modified external edges do not cross the external edges ending at nodes of classes 0 and $k-1$, and avoiding subtorus edges as much as possible.

In Figure 9, the modified external edges for each subtorus pair are drawn thicker. Note that since $k = 4$, in this figure there is no node u of class $3 \leq \text{class}(u) \leq k-2$; external edge connection of such nodes when $k \geq 5$ is identical to that described in Section 3.2 (see Figure 7). Therefore, by maximising the number of subtorus pairs as described previously, the impact of this refinement of the number of crossings is optimised.

Next, we establish the new upper bound on $cr(T(3,k))$ that is induced by this refined drawing method of a $T(3,k)$. Once again, we start by expressing α the total number of crossings that are induced by external edges. Yet, to the difference of the previous drawing methods, there are now crossings that involve two external edges (see Figure 9). All these crossings involving two external edges are induced by the modified external edges at each subtorus pair. For the

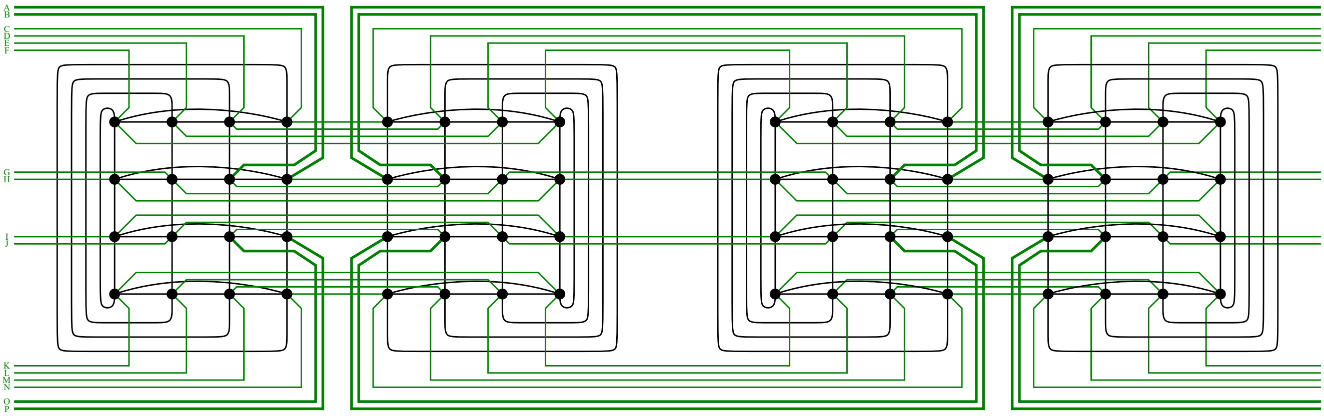


Fig. 9 Improved subtorus connection by relying on subtorus flipping. Here $k = 4$, so two subtorus pairs are induced.

sake of clarity when counting crossings, we say that for each subtorus pair, the crossings between the modified external edges and other external edges occur at T_f when the modified external edges cross the IN external edges for T_f. Similarly, the crossings between the modified external edges and other external edges occur at T_n when the modified external edges cross the OUT external edges for T_n. This way, we will count the crossings between external edges separately for each subtorus of a subtorus pair.

Define P and $\bar{P}$ the set of paired and unpaired subtori, respectively. Furthermore, let ϵ (resp. $\bar{\epsilon}$) be the total number of crossings induced at one paired (resp. unpaired) subtorus t by the external edges that end at a node of t. Hence, we have:

$$\alpha = \epsilon \cdot |P| + \bar{\epsilon} \cdot |\bar{P}|$$

The number of paired subtori is $|P| = k - (k \bmod 2)$ and that of unpaired subtori is $|\bar{P}| = k - |P| = k \bmod 2$.

First, we directly have:

$$\bar{\epsilon} = \beta = 2k^3 - 2k^2 - 2k$$

since the number of crossings induced by external edges remains unchanged in the case of an unpaired subtorus. Effectively, there is no modified external edge in the case of an unpaired subtorus.

Next, we establish ϵ. Let $\#_p C$ denote the total number of crossings induced at one paired subtorus t by the external edges that end at the nodes $u \in t$ with $\text{class}(u) \in C$. We have:

$$\epsilon = \#_p\{0\} + \#_p\{1\} + \#_p\{2, 3, \ldots, k-3\} + \#_p\{k-2\} + \#_p\{k-1\}$$

where this time five node classes are distinguished: $\{0\}$, $\{1\}$, $\{2, 3, \ldots, k-3\}$, $\{k-2\}$ and $\{k-1\}$. It should be noted that since, when two external edges cross each other, a modified external edge is necessarily involved, these external edge-only crossings are counted when considering the modified external edges, that is, included in $\#_p\{1\}$ and $\#_p\{k-2\}$. Furthermore, when considering the subtori of one subtorus pair, the respective subtorus drawings are exact replica, yet mirrored. Hence, to calculate ϵ, we can assume without loss of generality that the subtorus is non-flipped, which will simplify crossing counting.

Therefore, hereinafter, crossing counting is done considering a paired non-flipped subtorus.

Calculation of $\#_p\{0\}$

The number of crossings induced by the IN external edges is $(k-2) + \sum_{i=1}^{k}(k-i)$ and that induced by the OUT external edges is $\sum_{i=1}^{k}(k-i)$. Therefore, we have $\#_p\{0\} = k^2 - 2$.

Calculation of $\#_p\{1\}$

The IN external edges that end at the leftmost $\lceil k/2 \rceil$ nodes of this class induce the following number of crossings:

$$\underbrace{k\lceil k/2 \rceil}_{\substack{\text{wrap-around}\\\text{vertical}}} + \underbrace{\lceil k/2 \rceil - 1}_{\substack{\text{wrap-around}\\\text{horizontal}}} + \underbrace{\sum_{i=1}^{\lceil k/2 \rceil - 1} i}_{\text{internal vertical}}$$

$$= \lceil k/2 \rceil \left(k + \frac{\lceil k/2 \rceil + 1}{2}\right) - 1$$

The IN external edges that end at the rightmost $\lfloor k/2 \rfloor$ nodes of this class induce the following number of crossings:

$$\underbrace{\lfloor k/2 \rfloor - 1}_{\substack{\text{wrap-around}\\\text{horizontal}}} + \underbrace{k\lfloor k/2 \rfloor}_{\substack{\text{crossings by the}\\\text{modified external edges}}} + \underbrace{\sum_{i=1}^{\lfloor k/2 \rfloor - 1} i}_{\text{internal vertical}}$$

$$= \lfloor k/2 \rfloor \left(k + \frac{\lfloor k/2 \rfloor + 1}{2}\right) - 1$$

The number of crossings induced by the OUT external edges is as follows:

$$\underbrace{\sum_{i=1}^{k}(k-i)}_{\text{internal vertical}} = \frac{k^2 - k}{2}$$

Therefore, we have:

$$\begin{aligned}\#_p\{1\} &= \lceil k/2\rceil\left(k+\frac{\lceil k/2\rceil+1}{2}\right)-1\\ &\quad+\lfloor k/2\rfloor\left(k+\frac{\lfloor k/2\rfloor+1}{2}\right)-1+\frac{k^2-k}{2}\\ &= 2k^2-\lceil k/2\rceil\lfloor k/2\rfloor-2\end{aligned}$$

Calculation of $\#_p\{2, 3, \ldots, k-3\}$

First, for one such class: the number of crossings induced by the IN external edges is $k^2+(k-2)+\sum_{i=1}^{k}(k-i)$ and that induced by the OUT external edges is $\sum_{i=1}^{k}(k-i)$. Therefore, considering all these classes, we have $\#_p\{2,3,\ldots,k-3\} = (k-4)(k^2+(k-2)+\sum_{i=1}^{k}(k-i)+\sum_{i=1}^{k}(k-i)) = 2k^3-8k^2-2k+8$.

Calculation of $\#_p\{k-2\}$

By symmetry of $\#_p\{1\}$, we directly have: $\#_p\{k-2\} = 2k^2-\lceil k/2\rceil\lfloor k/2\rfloor-2$

Calculation of $\#_p\{k-1\}$

By symmetry of $\#_p\{0\}$, we directly have: $\#_p\{k-1\} = k^2-2$.

As a result, we have:

$$\epsilon = 2k^3-2k^2-2k-2\lceil k/2\rceil\lfloor k/2\rfloor$$

Thus,

$$\begin{aligned}\alpha &= (2k^3-2k^2-2k+2\lceil k/2\rceil\lfloor k/2\rfloor)(k-(k \bmod 2))\\ &\quad+\left(2k^3-2k^2-2k\right)(k \bmod 2)\\ &= 2k^4-2k^3-2k^2-2\lceil k/2\rceil\lfloor k/2\rfloor(k-(k \bmod 2))\end{aligned} \quad (3)$$

This final discussion regarding an upper bound on $cr(T(3,k))$ is summarised by Theorem 2.

Theorem 2. *The crossing number of a $T(3,k)$ satisfies the following relation:*

$$cr(T(3,k)) \le 2k^4-k^3-4k^2-2\lceil k/2\rceil\lfloor k/2\rfloor(k-(k \bmod 2))$$

Proof. As explained, the case $k=1$ induces a planar graph, and the case $k=2$ induces the relation $cr(T(3,2))=0$ due to [10], and thus a planar graph as well. Regarding the case $k \ge 3$, this can be easily derived from (1) and (3). □

The difference between the upper bound as derived in Section 3.1 (and similarly Section 3.2) and that of this section is thus as follows:

$$2k\lceil k/2\rceil\lfloor k/2\rfloor-2(k \bmod 2)$$

Not only does the difference obviously tend towards $+\infty$ as k increases, but it is also of cubic order, thus showing the significance of this improvement.

4. Deriving an upper bound on the crossing number of a $T(n,k)$

Establishing a tight upper bound on the crossing number of a torus $T(n,k)$ is difficult. Relying on a method similar to that of Section 3 is clearly impractical. In this section, we derive an upper bound on $cr(T(n,k))$ from the previously obtained upper bounds on $cr(T(2,k))$ and $cr(T(3,k))$.

We proceed recursively as follows. Consider a drawing of $T(n,k)$ that consists in non-overlapping k subtori $T(n-1,k)$. For each external edge (i.e., edge linking two nodes of distinct subtori $T(n-1,k)$), we consider an upper bound of the number of crossings involving this external edge at one subtorus $T(n-1,k)$. Then, this number of crossings is multiplied by the number of external edges that end at a node of one subtorus. Finally, the obtained number of crossings involving such an external edge is multiplied by k the number of $T(n-1,k)$ subtori. This drawing method is recursively applied until $n=4$, with thus $T(3,k)$ subtori. It is assumed in this drawing method that there is no crossing that involves two external edges. Obviously, while we derive a $T(n,k)$ drawing method and the corresponding number of crossings that is based on the drawing of k non-overlapping subtori $T(n-1,k)$, there is no guarantee that this drawing method is optimal, that is, can induce the crossing number $cr(T(n,k))$.

An upper bound δ of the crossing number that is induced by one external edge at one subtorus $T(n-1,k)$ can be safely set as the number of subtorus edges as follows:

$$\delta = ||T(n-1,k)|| = (n-1)k^{n-1}$$

Hence, since we assumed that external edges do not cross each other when $n \ge 4$, considering that there are k subtori $T(n-1,k)$ and that there are $2k^{n-1}$ external edges per subtorus $T(n-1,k)$, the upper bound of the crossing number C induced by external edges in a $T(n,k)$ is as follows:

$$C \le k \cdot 2k^{n-1} \cdot \delta$$

This discussion is summarised in Theorem 3 below.

Theorem 3. *The crossing number of a torus $T(n,k)$ satisfies the following recursive relation:*

$$\begin{aligned}&cr(T(n,1)) = 0\\ &cr(T(2,k)) \le k(k-2) && (k \ge 2)\\ &cr(T(3,2)) = 0\\ &cr(T(3,k)) \le 2k^4-k^3-4k^2\\ &\qquad -2\lceil k/2\rceil\lfloor k/2\rfloor(k-(k \bmod 2)) && (k \ge 3)\\ &cr(T(n,2)) \le 4^n \cdot 5/32-\lfloor(n^2+1)/2\rfloor 2^{n-2}\\ &cr(T(n,k)) \le k \cdot cr(T(n-1,k))\\ &\qquad +2(n-1)k^{2n-1} && (k \ge 3)\end{aligned}$$

Proof. As explained, the case $k=1$ induces a planar graph, and the case $k=2$ induces the relation $cr(T(3,2))=0$ and $cr(T(n,2)) \le 4^n \cdot 5/32-\lfloor(n^2+1)/2\rfloor 2^{n-2}$ due to [10]. Regarding the case $k \ge 3$, this can be easily derived from:

$$C \le k \cdot 2k^{n-1} \cdot \delta = 2(n-1)k^{2n-1}$$

and

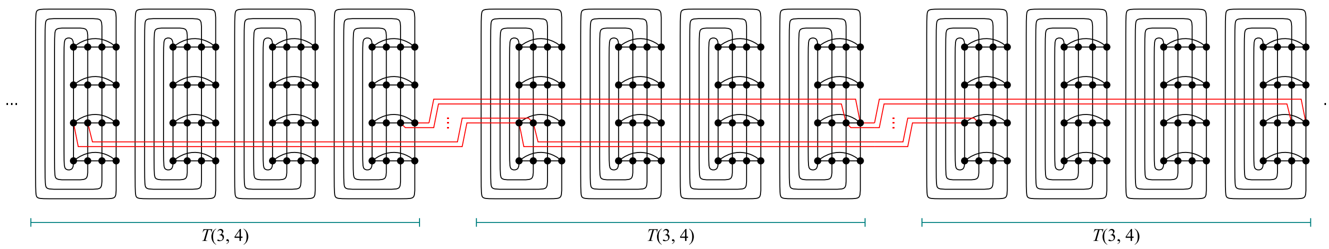


Fig. 10 Subtorus connection in the case of a $T(4,4)$. For the sake of clarity, the external edges of each $T(3,4)$ subtorus are omitted and only sample external edges of $T(4,4)$ are drawn, in red. Besides, only three (out of four) subtori $T(3,4)$ are represented.

$$cr(T(n,k)) \leq k \cdot cr(T(n-1,k)) + C$$

In addition, it has been assumed that when $n \geq 4$, there is no crossing that involves two external edges (i.e., two edges each linking two nodes of distinct subtori $T(n-1,k)$). Indeed, external edges can be drawn as described in Section 3.1, no matter the value of the dimension n. Without loss of generality, such a drawing in the case of a $T(4,4)$ is given in Figure 10 as an example. In this figure, for the sake of clarity, the external edges of each $T(3,4)$ subtorus (i.e., the green edges of Figure 9) are omitted and only sample external edges of $T(4,4)$ are drawn, in red. Besides, only three (out of four) subtori $T(3,4)$ are represented. We can see with this drawing scheme – we repeat that it is based on that of Section 3.1 – that no two external edges of $T(4,4)$ (i.e., in red) cross and that one external edge of $T(4,4)$ (i.e., in red) crosses at most once any non-red edge (i.e., any subtorus edge). The same drawing method can be applied for higher torus dimensions. Since the case $n = 3$ is one base case of the recursion, the fact that the drawing of a $T(3,k)$ includes external edge crossings does not contradict this statement. □

The following corollary is thus induced.

Corollary 1. *The crossing number of a $T(n,k)$ satisfies the following relation:*

$$cr(T(n,k)) = \begin{cases} O(n^2k^{2n-2}) & \text{if } n \geq k \\ O(nk^{2n-1}) & \text{otherwise} \end{cases}$$

Proof. From Theorem 3, we have:

$$\begin{aligned}
&cr(T(n,1)) = 0 = O(1) \\
&cr(T(2,k)) \leq k(k-2) = O(k^2) && (k \geq 2) \\
&cr(T(3,2)) = 0 = O(1) \\
&cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2 \\
&\qquad - 2\lceil k/2 \rceil \lfloor k/2 \rfloor (k - (k \bmod 2)) \\
&\qquad = O(k^4) && (k \geq 3) \\
&cr(T(n,2)) \leq 4^n \cdot 5/32 - \lfloor (n^2+1)/2 \rfloor 2^{n-2} = O(4^n) \\
&cr(T(n,k)) \leq k \cdot cr(T(n-1,k)) \\
&\qquad + 2(n-1)k^{2n-1} && (k \geq 3) \\
&\quad \leq k^2 \cdot cr(T(n-2,k)) + 2(n-2)k^{2n-2} \\
&\qquad + 2(n-1)k^{2n-1} \\
&\quad \leq k^3 \cdot cr(T(n-3,k)) + 2(n-3)k^{2n-4} \\
&\qquad + 2(n-2)k^{2n-2} + 2(n-1)k^{2n-1} \\
&\quad \vdots \\
&\quad \leq k^{n-3} \cdot cr(T(3,k)) + \left[2k^2 \sum_{i=3}^{n-2} ik^{2i} \right] \\
&\qquad + 2(n-1)k^{2n-1} \\
&\quad = k^{n-3}O(k^4) + O(n^2k^{2n-2}) + O(nk^{2n-1}) \\
&\quad = O(k^{n+1}) + O(n^2k^{2n-2}) + O(nk^{2n-1})
\end{aligned}$$

□

5. Conclusions

The crossing number problem is a difficult problem – NP-hard when solved for any graph. We have discussed in this paper the crossing number of a torus network. First, in addition to several trivial cases, we have discussed the crossing number of a 2-dimensional k-ary torus where $k \geq 2$, recalling that $cr(T(2,k)) \leq k(k-2)$, discussion which laid foundations for the rest of the paper. Second, we have considered the crossing number of a 3-dimensional k-ary torus and shown that it has an upper bound as follows: $cr(T(3,k)) \leq 2k^4 - k^3 - 4k^2 - 2\lceil k/2 \rceil \lfloor k/2 \rfloor (k - (k \bmod 2))$, which is a cubic order improvement compared to our previous work. Third, we have derived from these results an upper bound on the crossing number of an n-dimensional k-ary torus where $n \geq 4$ and $k \geq 3$:

$$cr(T(n,k)) \leq k \cdot cr(T(n-1,k)) + 2(n-1)k^{2n-1}$$

And, it has been shown that $cr(T(n,k))$ is $O(n^2k^{2n-2})$ when $n \geq k$ and $O(nk^{2n-1})$ otherwise.

Regarding future works, it would be interesting to establish a lower bound on $cr(T(3,k))$. In addition, trying to further refine the established upper bound on $cr(T(n,k))$ is meaningful especially as it was pessimistically estimated in this paper. Finally, while the obtained upper bound applies to any surface, including the plane, it could be interesting to investigate the existence of a better upper bound with a topology other than the plane.

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Antoine Bossard is an Associate Professor of the Graduate School of Science, Kanagawa University, Japan. He received the BS and MS degrees from Université de Caen Basse-Normandie, France in 2005 and 2007, respectively, and the Ph.D. degree from Tokyo University of Agriculture and Technology, Japan in 2011. His research is focused on graph theory, interconnection networks and dependable systems. For several years, he has also been conducting research regarding Chinese characters and their processing by computer systems. He is a member of ACM, ISCA and TUG.

Keiichi Kaneko is a Professor at Tokyo University of Agriculture and Technology. His main research areas are dependable systems, interconnection networks, functional programming, parallel and distributed computation, partial evaluation and educational systems. He received the B.E., M.E. and Ph.D. degrees from the University of Tokyo in 1985, 1987 and 1994, respectively. He is a member of IEEE, ACM, IEICE, IPSJ and JSSST.

Frederick C. Harris, Jr. received his BS and MS degrees in Mathematics and Educational Administration from Bob Jones University, Greenville, SC, USA in 1986 and 1988 respectively. He then went on and received his MS and Ph.D. degrees in Computer Science from Clemson University, Clemson, SC, USA in 1991 and 1994 respectively. He is currently a Professor in the Department of Computer Science and Engineering and the Director of the High Performance Computation and Visualization Lab at the University of Nevada, Reno, USA. He is also the Nevada State EPSCoR Director and the Project Director for Nevada NSF EPSCoR. He has published more than 290 peer-reviewed journal and conference papers along with several book chapters. He has had 14 Ph.D. students and 80 MS Thesis students finish under his supervision. His research interests are in parallel computation, simulation, computer graphics, and virtual reality. He is also a Senior Member of the ACM, and a Senior Member of the International Society for Computers and their Applications (ISCA).