Word-smithing is a much greater percentage of what I am supposed to be doing in life than I would ever have thought.
DONALD KNUTH [**6**, p. 54]

Writing a Math Phase Two Paper

STEVEN L. KLEIMAN
with the collaboration of GLENN P. TESLER
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Abstract. We discuss the kind of writing that's appropriate in a paper submitted to the math department to complete Phase Two of MIT's writing requirement. First, we review the general purpose of the requirement and the specific way of completing it for the math department. Then we consider the writing itself: the organization into sections, the use of language, and the presentation of mathematics. Finally, we give a short example of mathematical writing.

1. Introduction. MIT established the writing requirement to ensure that its graduates can write both a good general essay and a good technical report. Correspondingly, the requirement has two phases, which engage students at the beginning and toward the end of their undergraduate careers. The requirement is governed by an institute committee, the Committee on the Writing Requirement (CWR). The requirement is administered by the Undergraduate Education Office, which works in cooperation with the individual departments on Phase Two. The general information given here about the requirement is taken from the MIT *Bulletin* and the CWR's brochure [**3**], which are the official sources.

To complete Phase One, students must achieve a suitable score on the College Board Achievement Test or Advanced Placement Examination, pass the Freshman Essay Evaluation, pass an appropriate writing subject in Course 21 and be certified by the instructor, or write a satisfactory five page paper for any MIT subject, Wellesley exchange subject, or UROP activity. In level, format, and style, a Phase One paper should be like a magazine article for an informed, but general, readership. Papers are judged on their logical structure, language and tone, technical accuracy, and mechanics (grammar, spelling, and punctuation) by the instructor of the subject and by evaluators for the Undergraduate Education Office. A paper judged not acceptable may be revised and resubmitted twice. Students must complete Phase One by the middle of their third semester at the Institute.

To complete Phase Two, students must receive a grade of B or better for the quality of writing in a cooperative subject approved by the student's major department, receive a grade of B or better in one of several advanced classes in technical writing, or write a satisfactory ten page paper for any MIT subject or UROP activity approved by the major department. A student with two majors needs only to complete the requirement in one department. In level, format, and style, a Phase Two paper should be like a formal professional report. Thus a term paper or laboratory report may have to be reworked substantially before it is acceptable as a Phase Two paper. The paper is judged by its supervisor primarily for the technical content and by departmental evaluators primarily for the quality of writing. Students must complete Phase Two by the end of registration day of their last semester; otherwise, they must petition their departments and the CWR. Petitions for permission to enroll in a writing subject are routinely approved; petitions to submit a late paper are approved only when there are exceptional circumstances.

In the Department of Mathematics, there is no cooperative subject, and most students write a paper to satisfy Phase Two. Between one-half and three-quarters of the papers begin as term papers for one course, 18.310 Principles of Applied Mathematics. Every paper must have some technical mathematics in it. When the student and the supervisor feel the paper is ready, the student picks up a cover sheet in the Undergraduate Mathematics Office, Room 2–108. The student fills out the top, and gives it to the supervisor, who must vouch for the paper's technical accuracy, and may comment on the quality of the writing. The student then submits the paper and the cover sheet to the departmental coordinator. The paper must be submitted by the start of IAP if the student intends to graduate the following June. After a paper is submitted, it is read for organization and language by the coordinator, who determines whether the paper is acceptable as is or needs improvement. If the paper needs improvement, the department's Writing TA contacts the student and sets up an appointment to discuss the areas requiring further work. The student submits further revisions to the TA, and when the paper is perfected, it is resubmitted to the coordinator. On occasion, the coordinator works directly with the student. Thus, not only is the paper improved, but, more importantly, the student becomes a better writer. The process is not examinational, but tutorial.

The present paper is a primer on mathematical writing, especially the writing of short papers. Indeed, this paper itself is intended to be a model of format and style. Mathematical writing is primarily a craft, and anyone can learn it. The aim is to inform efficiently. The basic principles are discussed and illustrated here. Some of these principles are simple matters of common sense; others are conventions that have evolved from experience. None need be followed slavishly, but none should be broken thoughtlessly. When one is broken, the break may stand out like a sore thumb — just as unconventional spelling does. However, the writing itself should fade into the background, leaving the information to be conveyed out front. Following these principles will not cramp anyone's style; there's plenty of room for individual variation. The various principles are discussed more fully in a number of works, including the following works on which this primer is based: Alley's down-to-earth book [**1**], Flanders' article [**4**] and Gillman's manual [**5**] for authors of articles for MAA journals, the notes [**6**] to Knuth's Stanford course on mathematical writing, and Munkres' brief manual of style [**7**].

In Section 2, we discuss the normal way a short mathematical paper is broken into sections. We consider the purpose and content of the individual sections: the abstract, the introduction, the several sections of the main discussion, the conclusion (which is rare in a mathematical work), the appendix, and the list of references. In Section 3, we deal with "language," that is, the choice of words and symbols, and the structuring of sentences and paragraphs. We consider seven goals of language: precision, clarity, familiarity, forthrightness, conciseness, fluidity, and imagery. We discuss the meaning of these goals and how best to meet them. Sections 2 and 3 are based mainly on Alley's book [**1**]. In Section 4, we deal with a number of special problems that arise in writing mathematics, such as the treatment of formulas, the presentation of theorems and proofs, and the use of symbols. The material is drawn from all five sources cited above. In Section 5, we give an illustrative sample of mathematical writing. We treat the two fundamental theorems of calculus, for the most part paraphrasing the treatment in Apostol's book [**2**, pp. 202–4]; we state and prove the theorems, and explain their significance. Finally, in the appendix, we deal with the use of such terms as lemma, proposition, and definition, which are common in treatments of advanced mathematics.

2. Organization.

Most short technical papers are divided up into about a half dozen sections, which are numbered and titled. (The pages too should be numbered for easy reference.) Most papers have an abstract, an introduction, a number of sections of discussion, and a list of references. On occasion, papers have appendices, which give special detailed information, or provide necessary general background to secondary audiences. In mathematics, few papers have a section of conclusions and recommendations. Such a section would discuss the results from an overall perspective, bring together the loose ends, and possibly make recommendations for future research. In mathematical papers, these issues are almost always incorporated into the introduction. Normally, short papers have no formal table of contents.

Sectioning involves more than merely dividing up the material; you have to decide about what to put where, about what to leave out, and about what to emphasize. If you make the wrong decisions, you will lose your readers. There is no simple formula for deciding, because the decisions depend heavily on the subject and the audience. However, you must structure your paper in a way that is easy for your readers to follow, and you must emphasize the key results.

The title is very important. If it is inexact or unclear, it will not attract all the intended readers. A strong title identifies the general area of the subject and its most distinctive features. A strong title contains no distracting secondary details and no formulas. A strong title is concise.

The abstract is the most important section. First it identifies the subject; it repeats words and phrases from the title to corroborate a reader's first impression, and it gives details that didn't fit into the title. Then it lays out the central issues, and summarizes the discussion to come. It is drawn completely from the paper. However, it includes no general background material. The abstract is a table of contents in a paragraph of prose. It allows readers to decide quickly about reading on. While many will decide to stop there, the potentially interested will continue. The goal is not to entice all, but to inform the interested efficiently. Remember, readers are busy. They have to decide quickly whether your paper is worth their time. They have to decide whether the subject matter is of interest to them, and whether the presentation will bog them down. A well-written abstract will increase the readership.

The introduction is the place where readers settle into the "story," and often make the final decision about reading the whole paper. Start strong; don't waste words or time. Your readers have just read your title and abstract, and they've gained a general idea of your subject and treatment. However, they are probably still wondering what exactly your subject is and how you'll present it. A strong introduction answers these questions with clarity and precision. It identifies the subject precisely and instills interest in it by giving details that did not fit into the title or abstract, such as how the subject arose and where it is headed, how it relates to other subjects and why it is important. A strong introduction touches on all the significant points, and no more. A strong introduction gives enough background material for understanding the paper as a whole, and no more. Put background material pertinent to a particular section in that section, weaving it unobtrusively into the text. A strong introduction discusses the relevant literature, citing a good survey or two. Finally, a strong introduction describes the organization of the paper, making explicit references to the section numbers. It summarizes the contents in more detail than the abstract, and it says what can be found in each section. It gives a "road map," which indicates the route to be followed and the prominent features along the way. This road map is placed at the end of the introduction to ease the transition into the next section.

The body discusses the various aspects of the subject individually. In writing the body, your hardest job is developing a strategy for parceling out the information. Every paper requires its own strategy, which must be worked out by trial and error. There are, however, a few guidelines. First, present the material in small digestible portions. Second, beware of jumping haphazardly from one detail to another, and of illogically making some details specific and others generic. Third, if possible, follow a sequential path through the subject. If such a path simply doesn't exist, then break the subject down into logical units, and present them in the order most conducive to understanding. If the units are independent, then order them according to their importance to the primary audience.

There are three main reasons for dividing the body into sections: (1) the division indicates the strategy of your presentation; (2) it allows readers to quickly and easily find the information that interests them; and (3) it gives readers restful white space, allowing them to stop and reflect on what was said. Make the introduction and the several sections of the body roughly equal in length. When you title a section, strive for precision and clarity; then readers will have an easier time finding particular information. Don't simply insert a title, as is often done in newspaper articles, to recapture interest; rather, wind down the discussion in the first section in preparation for a break, and then restart the discussion in the next section, after the title. When you refer to Section 3, remember to capitalize the word "Section"; it is considered a proper name. Don't use subsections in a short paper; they make the flow too choppy.

Each main point should be accented via stylistic repetition, illustration, or language. Stylistic repetition is the selective repetition of something important; for example, you should talk about the important points once in the abstract, a second time in the introduction, and a third time in the body. When appropriate, repeat an important point in a figure or diagram. Finally, accent an important point with a linguistic device: italics, boldface, or quotations marks; a one sentence paragraph; or a short sentence at the end of a long paragraph. In particular, set a technical term in italics or boldface — or enclose it in quotation marks if it is only moderately technical — once, at the time it is being defined. Do not underline when italics or boldface is available. Use headings such as **Table 1-1**, **Figure 1-2**, and **Theorem 5-2**, and refer to them as Table 1-1, Figure 1-2, and Theorem 5-2; note that the references are capitalized and set in roman. When you employ linguistic devices, be consistent: always use the same device for the same job.

The list of references contains bibliographical information about each source cited. The style of the list is different in technical and nontechnical writing; so is the style of citation. In fact, there are several different styles used in technical writing, but they are relatively minor variations of each other. The style used in this paper is commonly used in contemporary mathematical writing. The citation is treated somewhat like a parenthetical remark within a sentence. Footnotes are not used; neither are the abbreviations "loc. cit.," "op. cit.," and "ibid." The reference key, usually a numeral, is enclosed in square brackets. Within the brackets and after the reference key, place — as a service — specific page numbers, section numbers, or equation numbers, preceded by a comma [**5**, Section 3.7, p. 9]. If the citation comes at the end of a sentence, put the period after the citation, not before the brackets or inside them.

3. Language. In the subject of writing, the word "language" means the choice of words and symbols, and their arrangement in phrases. It means the structuring of sentences and paragraphs, and the use of examples and analogies. When you write, watch your language. When it falters, your readers stumble; if they stumble too often, they'll lose their patience and stop reading. Write, rewrite, then rewrite again, improving your language as you go; there is no short cut!

Alley [**1**, pp. 25–130] identifies seven goals of language: two primary goals — precision and clarity — and five secondary goals — familiarity, forthrightness, conciseness, fluidity, and imagery. These goals often reinforce one another. For example, clarity and forthrightness promote conciseness; precision and familiarity promote clarity. We will now consider these goals individually.

Being precise means using the right word. However, finding the right word can be difficult. Consult a dictionary, not a thesaurus, because the dictionary explains the differences among words. For example, the *American Heritage Dictionary* is a good choice, because it has many notes on usage. Consult a book on usage, such as *Webster's Dictionary of English Usage*. Always consider a word's connotations (associated meanings) along with its denotations (explicit meanings); the wrong connotations can trip up your readers by suggesting unintended ideas. For example, the word "adequate" means enough for what is required, but it gives you the feeling that there's not quite enough; its connotation is the exact opposite of its denotation. Strong writing does not require using synonyms, contrary to popular belief. Indeed, by repeating a word, you often strengthen the bond between two thoughts. Moreover, few words are exact synonyms, and often, using an exact synonym adds nothing to the discussion.

Being precise means giving specific and concrete details. Without the details, readers stop and wonder needlessly. On the other hand, readers remember by means of the details. Being precise does not mean giving all the details, but giving the informative details. Giving the wrong details or giving the right ones at the wrong time makes the writing boring and hard to follow. Being specific does not mean eradicating general statements. General statements are important, particularly in summaries. However, specific examples, illustrations, and analogies add meaning to the general statements.

Being clear means using no wrong words. An ambiguous phrase or sentence will disrupt the continuity and diminish the authority of an entire section. A common mistake is to use overly complex prose. Don't string adjectives before nouns, lest they lose their strength and precision; instead, use prepositional phrases and dependent clauses, or use two sentences. Keep your sentences simple and to the point. It may help to keep most of them short, but you need some longer ones to keep your writing from sounding choppy and to provide variety and emphasis.

A pronoun normally refers to the previous noun. Unfortunately, it is common to abuse pronouns, particularly "it," "this," and "which." Make sure the reference is immediately clear, especially when you refer broadly to a preceding phrase, topic, or idea. It is also common to use a plural pronoun such as "their" to refer back to a singular, but indefinite, antecedent such as a "reader." This usage is still considered unacceptable in formal writing; reformulate your sentence if necessary. The pronouns "that" and "which" are not always interchangeable. Either may be used to introduce a restrictive clause, but use "that" ordinarily. Only "which" may be used to introduce a descriptive clause, and the clause must be set off with commas. Strunk and White in their classic guide to style [**8**, p. 47] recommend "which-hunting."

Punctuation is used to eliminate ambiguities in language, and to smooth the flow of the text. A simple misuse of punctuation can weaken your authority. Learn how to punctuate properly, and use a handbook like *The Chicago Manual of Style.* In optional cases, use the punctuation if it promotes clarity at all, but strive for consistency through out the paper. Here are a few rules.

Use periods only to end sentences. (A complete sentence within parentheses should begin with a capital letter and end with a punctuation mark, unless the sentence is part of another and would end with a period.) Avoid abbreviations that require periods; for example, use "that is" instead of "i.e." Always use commas to separate three or more items in a list and to set off contrasted elements (they often begin with "but" or "not"). Most of the time, use a comma after an introductory phrase. Use colons to introduce lists, explanations, and displays. Do not use colons in continuing statements: if a statement is stopped at the colon, then the words should form a complete sentence. Use a semicolon to join two sentences to indicate that they are closely linked in content; however, if you insert a conjunction (not an adverb), then use a comma. Use a dash as a comma of extra strength — but use it sparingly. Place closing quotation marks (") after commas and periods; it is a matter of appearance, not logic. Enclose parenthetical material in parentheses; footnotes and endnotes are generally discouraged in technical reports.

To inform, you must use language familiar to your readers. Define unfamiliar words, and familiar words used in unfamiliar ways. If the definition is short, then include it in the same sentence, preceding it by "or" or setting it off by commas or parentheses. If the definition is complex or technical, then expand it in a sentence or two. Do not use words like "capability," "utilize," and "implement"; they offer no precision, clarity, or continuity and smack of pseudo-intellectualism. Beware of words like "interface"; they are precise in some contexts, yet imprecise and pretentious in others.

Jargon is vocabulary particular to a certain group, and it consists of abbreviations and slang terms. Jargon is not inherently bad. Indeed, it is useful in internal memos and reports. However, jargon alienates external readers and may even mislead them. So beware. Clichés are figurative expressions that have been overused and have taken on undesirable connotations. Most are imprecise and unclear. Avoid them, or be laughed at. In addition, avoid numerals because they slow down the reading. Write numbers out if they can be expressed in one or two words and are used as adjectives, unless they are accompanied by units, a percentage sign, or a monetary sign. For instance, write, "The equation has two roots," and "One root is 2." Don't begin a sentence with a numeral or a symbol; reformulate the sentence if necessary.

Be forthright: write in an unhesitating, straightforward, and friendly style, ridding your language of needless and bewildering formality. Be wary of awkward and inefficient passive constructions. Often the passive voice is used simply to avoid the first person. However, the pronoun "we" is now generally considered acceptable in contexts where it means the author and reader together, or the author with the reader looking on. (Still, "we" should not be used as a formal equivalent of "I," and "I" should be used rarely, if at all.) For instance, don't write, "By solving the equation, it is found that the roots are real." Instead write, "Solving the equation, we find the roots are real," or "Solving the equation yields *real* roots." It is acceptable, but less desirable, to write, "Solving the equation, one finds the roots are real"; the personal pronoun "one" is a sign of formality, and is uncommon in mathematical writing today. Beware of dangling participles. It is wrong to write, "Solving the equation, the roots are real," because "the roots" cannot solve the equation.

Concise writing is vigorous. Conciseness comes from reducing sentences to their simplest forms. For instance, don't write, "In order to find the solution of the equation, we can use one of two alternative methods." Instead, write, "To solve the equation, we can use one of two methods," thus eliminating empty words ("in order to"), reducing fat phrases ("to find the solution of"), and eliminating needless repetition ("alternative"). Concise writing is simple and efficient, thus beautiful.

The flow of a paper is disturbed by weak transitions between sentences and paragraphs. To smooth the flow, start a sentence where the preceding one left off. Use connective words and phrases. Avoid gaps in the logic, and give ample details. Don't take needless jumps when deriving equations. Use parallel wording when discussing parallel concepts. Don't raise questions implicitly, and leave them unanswered. Pay attention to the tense, voice, and mode of verbs.

Many papers stagnate because they lack variety. The sentences begin the same way, run the same length, and are of the same type. The paragraphs have the same length and structure. Don't worry about varying your sentences and paragraphs at first; wait until you polish your writing. Remember though, if you have to choose between fluidity and clarity, then you must choose clarity.

The very structure of a sentence conveys meaning. Readers expect the stress to lie at the beginning and end. They take a breath at the beginning, but will run out of breath before the end if the structure is too complex, for instance, if the subject is too far from the verb.

Most people think and remember images, not abstractions, and images are clarified by illustrations. Illustrations also give readers rest stops, so complex ideas can soak in. Moreover, illustrations can make a paper more palatable and less intimidating. However, the use of illustrations can be overdone; it must fit the audience and the subject.

Illustrations cannot stand alone; they must be introduced in the text. Assign them titles, like Figure 5-1 or Table 5-1, for reference. Assign them captions that tell, independently of the text, what they are and how they differ from one another, without being overly specific. In addition, clearly label the parts of your illustrations: label the axes of graphs with words, not symbols; identify any unusual symbols of your diagrams in the text. Don't put too much information into one illustration, because papers without white space tire readers. For the same reason, use adequate borders. Smooth the transitions between your words and pictures. First of all, match the information in your text and illustrations. Secondly, place the illustrations closely after — never before — their references in the text.

4. Mathematics. There are some special problems in mathematical writing because it tends to involve many abstract symbols and formal arguments. Here are some guidelines to keep in mind.

Formulas are difficult to read because readers have to stop and work through the meaning of each term. Don't merely list a sequence of formulas with no discernible goal, but give a running commentary. Define all the terms as they are introduced, state any assumptions about their validity, and give examples to provide a feeling for them. Similarly, motivate and explain formal statements. Don't simply call a statement "important," "interesting," or "remarkable," but explain why it is so.

Display an important formula by centering it on a line by itself, and give a reference number in the margin if it's especially important or if you need to refer to it. Also display any formula that's more than a quarter of a line long or that would be broken badly between lines. Punctuate the display with commas, a period, or whatever as you would if you had not displayed it.

Be clear about the status of every assertion; indicate whether it is a conjecture, the previous theorem, or the next corollary. If it is not a standard result and you omit its proof, then give a reference, preferably in the text just before the statement (If you give the reference in the statement, then do so after the heading like this: **Theorem 5-1** [**2**, p. 202].) Tell whether the omitted proof is hard or easy to help readers decide whether to try to work it out for themselves. If the theorem has a name, use it: say "by the First Fundamental Theorem," not "by Theorem 5-1." State a theorem before proving it. Keep the statement concise; put definitions and discussion elsewhere.

Prefer a conceptual proof to a computational one; ideas are easier to communicate, understand, and remember. Omit the details of purely routine computations and arguments — ones with no unexpected tricks and no new ideas. Beware of any proof by contradiction; often there's a simpler direct argument. Finally, when the proof has ended, say so outright; for instance, say, "The proof is now complete." In addition, surround the proof — and the statement as well — with some extra white space.

Here are some more guidelines:

1. Separate symbols in different formulas with words.

   Bad: Consider S_q, $q = 1, \dots, n$.
   Good: Consider S_q for $q = 1, \dots, n$.

2. Don't use such symbols as $\exists$, $\forall$, $\wedge$, $\Rightarrow$, $\approx$, $=$, $>$ in text; replace them by words. They may, of course, be used in formulas placed in text.

   Bad: Let S be the set of all numbers of absolute value < 1.
   Good: Let S be the set of all numbers of absolute value less than 1.
   Good: Let S be the set of all numbers x such that $|x| < 1$.

3. Don't start a sentence with a symbol.

   Bad: $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geq 0$.

   Good: The quadratic equation $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geq 0$.

4. Beware of using symbols to convey too much information all at once.

   Very bad: If $\Delta = b^2 - 4ac \geq 0$, then the roots are real.

   Bad: If $\Delta = b^2 - 4ac$ is nonnegative, then the roots are real.

   Good: Set $\Delta = b^2 - 4ac$. If $\Delta \geq 0$, then the roots are real.

5. If you introduce a condition with "if," then introduce the conclusion with "then."

   Very bad: If $\Delta \geq 0$, $ax^2 + bx + c = 0$ has real roots.

   Bad: If $\Delta \geq 0$, the roots are real.

   Good: If $\Delta \geq 0$, then the roots are real.

6. Don't set off by commas any symbol or formula used in text in apposition to a noun.

   Bad: If the discriminant, Δ, is nonnegative, then the roots are real.

   Good: If the discriminant Δ is nonnegative, then the roots are real.

7. Use consistent notation. Don't say "A_j where $1 \leq j \leq n$" one place and "A_k where $1 \leq k \leq n$" another.

8. Keep the notation simple. For example, don't write "x_i is an element of X" if "x is an element of X" will do.

9. Precede a theorem, algorithm, and the like with a complete sentence.

   Bad: We now have the following
   Theorem 4-1. $H(x)$ is continuous.

   Good: We can now prove the following result.
   Theorem 4-1. The function $H(x)$ defined by Formula (4-1) is continuous.

5. Example. As an example of mathematical writing, we discuss the two fundamental theorems of calculus. Our discussion is based on that in Apostol's book [**2**, pp. 202–7]. The First Fundamental Theorem says that the process of differentiation reverses that of integration. This statement is remarkable because the two processes appear to be so different: differentiation gives us the slope of a curve; integration, the area under the curve. Here is a precise statement of the theorem.

Theorem 5-1 (First Fundamental Theorem of Calculus). *Let f be a function defined and continuous on the closed interval $[a, b]$ and let c be in $[a, b]$. Then for each x in the open interval (a, b), we have*

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x).$$

Proof. Take a positive number h such that $x + h \leq b$. Then

$$\int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

By hypothesis, f is continuous. Hence there is some z in $[x,\, x+h]$ for which this last integral is equal to $h\,f(z)$ by the Mean Value Theorem for integrals [**2**, p. 154] (which is not hard to derive from the Intermediate Value Theorem). (The setup is shown in Figure 5-1; the Mean Value Theorem says that the area under the graph of f is equal to the area of the rectangle.) Therefore,

$$\frac{1}{h}\left(\int_c^{x+h} f(t)\,dt - \int_c^{x} f(t)\,dt\right) = f(z).$$

Now, $x \le z \le x+h$. Hence, as h approaches 0, the difference quotient on the left approaches $f(x)$. A similar argument holds for negative h. Thus the derivative of the integral exists and is equal to $f(x)$. The proof is now complete.

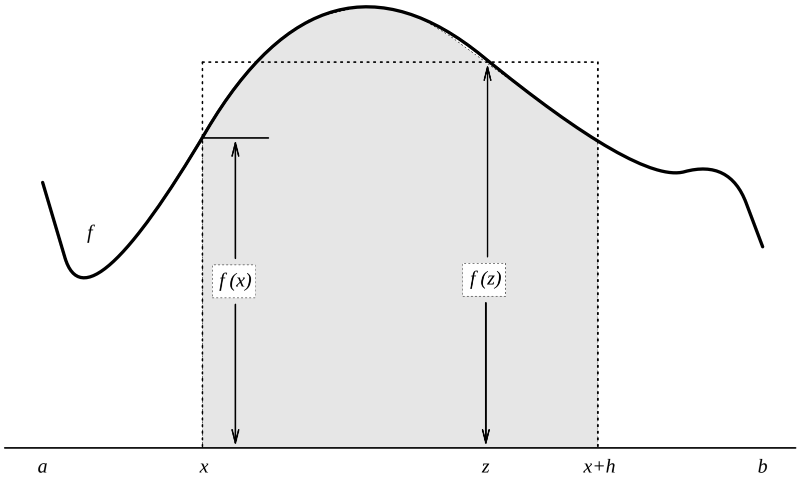

Figure 5-1. Geometric setup of the proof of the First Fundamental Theorem.

The First Fundamental Theorem says that, given a continuous function f, there exists a function F (namely, $F(x) = \int_c^x f(t)\,dt$) whose derivative is equal to f:

$$F'(x) = f(x).$$

Such a function F is called an *integral* (or a *primitive* or an *antiderivative*) of f. Integrals are not unique: if F is an integral of f, then obviously so is $F + C$ for any constant C. On the other hand, there is no further ambiguity: any two integrals F and G of f differ by a constant, because their difference $F - G$ has vanishing derivative,

$$(F - G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \text{ for every } x,$$

and hence $F - G$ is constant as a simple consequence of the Mean Value Theorem for derivatives (see [**2**, Theorem 4.7(c), p. 187]).

When we combine the First Fundamental Theorem with the fact that an integral is unique up to an additive constant, we obtain the following theorem.

Theorem 5-2 (Second Fundamental Theorem of Calculus). *Let f be a function defined and continuous on the open interval I, and let F be an integral of f on I. Then for each c and x in I,*

$$\int_c^x f(t)\,dt = F(x) - F(c). \tag{5-1}$$

Proof. Set $G(x) = \int_c^x f(t)\,dt$. By the First Fundamental Theorem, G is an integral of f. Now, any two integrals differ by a constant. Hence $G(x) - F(x) = C$ for some constant C. Taking $x = c$ yields $-F(c) = C$ because $G(c) = 0$. Thus $G(x) - F(x) = -F(c)$, and Equation (5-1) follows. The proof is now complete.

The Second Fundamental Theorem is a powerful statement. It says that we can compute the value of a definite integral merely by subtracting two values of any integral of the integrand. In practice, integrals are often found by reading a differentiation formula in reverse. For example, the integrals in Table 5-1 were found in this way. The notation in the table is standard [**9**, p. 178]: the equation

$$\int f(x)\,dx = F(x) + C$$

is read, "The integral of $f(x)\,dx$ is equal to $F(x)$ plus C."

Table 5-1
A brief table of integrals

1.	$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$, if $a \neq -1$
2.	$\int x^{-1}\,dx = \ln x + C$
3.	$\int \sin x\,dx = -\cos x + C$
4.	$\int \cos x\,dx = \sin x + C$
5.	$\int e^x\,dx = e^x + C$

A longer table of integrals is found on the endpapers of the calculus textbook [**9**].

Appendix. Advanced mathematics

In many treatments of advanced mathematics, the key results are stated formally as theorems, propositions, corollaries, and lemmas. However, these four terms are often used carelessly, robbing them of some useful information they have to convey: the nature of the result.

A *theorem* is a major result, one of the main goals of the work. Use the term "theorem" sparingly. Call a minor result a *proposition* if it is of independent interest. Call a minor result a *corollary* if it follows with relatively little proof from a theorem, a proposition, or another corollary. Sometimes a result could properly be called either a proposition or a corollary. If so, then call it a proposition if it relatively important, and call it a corollary if it is relatively unimportant. Call a subsidiary statement a *lemma* if it is used in the proof of a theorem, a proposition, or another lemma. Thus a lemma never has a corollary, although a lemma may be used, on occasion, in deriving a corollary. Normally, a lemma is stated and proved before it is used.

The terms "definition" and "remark" are also often abused. A formal *definition* should simply introduce some terminology or notation; there should be no accompanying discussion of the new terms or symbols. A formal *remark* should be a brief comment made in passing; the main discussion should be logically independent of the content of the remark. Often it is better to weave definitions and remarks into the general discussion rather than setting them apart formally.

Typographically, the statements of theorems, propositions, corollaries, and lemmas are traditionally set in italics, and the headings themselves are set in boldface or in caps and small caps (**Theorem** or THEOREM, and so forth). The texts of definitions and remarks are set as ordinary text; so are the texts of proofs, examples, and the like. These headings are traditionally set in italics, boldface, or small caps. (There is also a tradition of treating definitions typographically like theorems, but this tradition is less common today and less desirable.) All these formal statements and texts are usually set off from the rest of the discussion by putting some extra white space before and after them.

Assign sequential reference numbers to these headings, irrespective of their different natures, and use a hierarchical scheme whose first component is the section number. Thus "Corollary 3-6" refers to the sixth prominent statement in Section 3, and indicates that the statement is a corollary. If the statement is the second corollary of the third proposition in the paper, then it may be more logical to name the statement "Corollary 2," but doing so may make the statement considerably more difficult to locate.

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